

# Repeated Fair Allocation of Indivisible Items

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## Abstract

In practice, items are not always allocated once and for all, but often repeatedly. For example, when the items are recurring chores to distribute in a household. Motivated by this, we initiate the study of the **repeated fair division of indivisible items**.

## Applications

- Fairly distributing household chores between a couple
- Allocating teaching duties to professors over the semesters
- Granting employees daily access to a common infrastructure

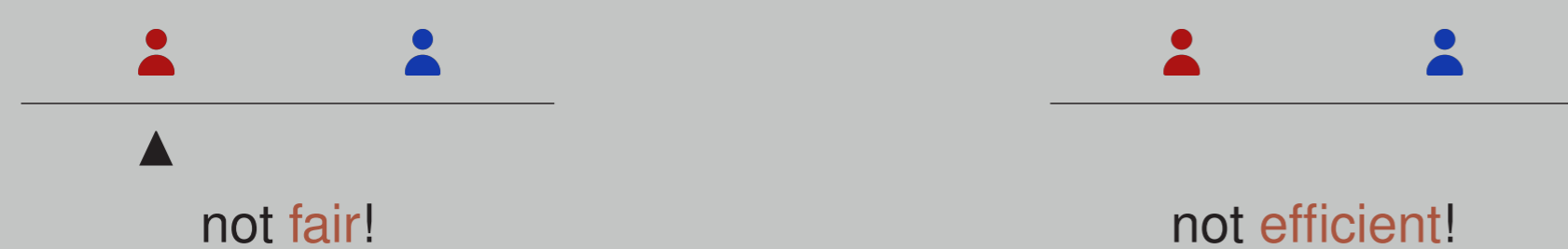
## Repetition: Why Bother?

In the one-shot setting, a **Proportional** (let alone Envy-Free) and **Pareto-Optimal** allocation may not exist. Our main goal:

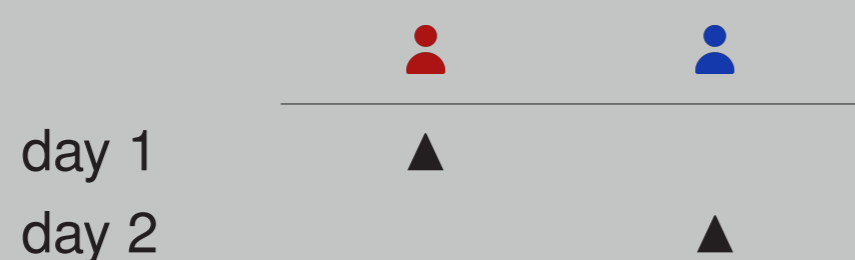
“Can we guarantee **better fairness** and **efficiency** properties by looking at the **repeated allocation** of items?”

## Main Idea

Suppose that we want to allocate a single item  $\blacktriangle$  between two agents,  $\color{red}{\bullet}$  and  $\color{blue}{\bullet}$ . Problem:



What if we share them **over time**?



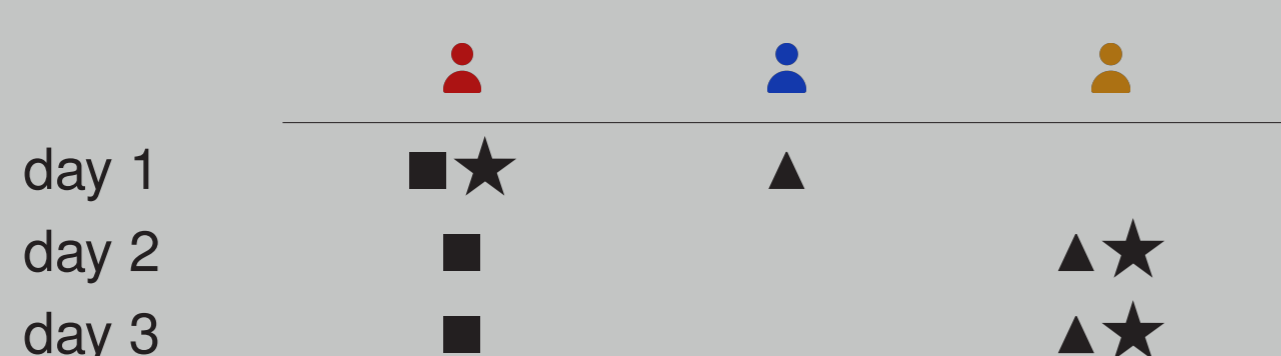
Each day's allocation is not fair, but the **overall allocation** is!

## Formal Model

We have  $n$  agents ( $\color{red}{\bullet}$ ,  $\color{blue}{\bullet}$ ,  $\color{yellow}{\bullet}$ , ...) that need to share some items ( $\blacktriangle$ ,  $\blacksquare$ ,  $\star$ , ...). Agents have additive utilities:

	$\color{red}{\bullet}$	$\color{blue}{\bullet}$	$\color{yellow}{\bullet}$
$\blacktriangle$	1	3	4
$\blacksquare$	5	2	1
$\star$	-3	-4	-2

We have  $k$  time-steps at our disposal. Example ( $k = 3$ ):



## Axioms

An axiom can be satisfied **overall** (while looking globally at the whole bundle, over all time-steps) or **per round** (if it is satisfied individually by all time-steps).

- Envy-freeness (EF)**: No agent prefers someone else's bundle
- Envy-freeness up to one item (EF1)**: If an agent envies some other agent, we can eliminate envy by removing one item from the bundle of one of the two agents
- Proportionality (PR)**: Each agent receives at least  $1/n$  of their evaluation of the whole set of items
- Pareto-optimality (PO)**: There is no reallocation that is strictly better for some agents, and worse for none

## Results: General Case

Under certain conditions, envy-freeness is always achievable:

*If  $k$  is a multiple of  $n$ , an **overall EF** allocation always exists.*

To achieve this, we can rotate the items at each time-step, e.g.:

	$\color{red}{\bullet}$	$\color{blue}{\bullet}$	$\color{yellow}{\bullet}$
day 1	$\blacktriangle$	$\blacksquare$	$\star$
day 2	$\star$	$\blacktriangle$	$\blacksquare$
day 3	$\blacksquare$	$\star$	$\blacktriangle$

What about efficiency? Even if  $k$  is a multiple of  $n$ , an overall EF and PO allocation **might not exist**. Still:

*If  $k$  is a multiple of  $n$ , an **overall PR and PO** allocation always exists.*

## Results: Two-agent Case

For two agents, we have stronger fairness guarantees:

*For two agents, if  $k$  is even, an **overall EF and PO** allocation always exists.*

What about the individual time-steps? We cannot have envy-freeness in every round. However:

*For two agents, if  $k$  is even, an allocation which is **overall EF and EF1 per round** always exists.*

Can we additionally have efficiency? **Not if  $k > 2$** , but:

*For two agents, if  $k = 2$ , we can always find an **overall EF and PO** allocation that is **EF1 per round**.*

*For two agents, if  $k$  is even, we can always find an **overall EF and PO** allocation that is **weakly EF1 per round**.*

## Results: Variable Number of Rounds

What if the number of rounds is not known in advance? Via a connection to the randomised and divisible settings, we show:

*For every utility profile, **there is some  $k$**  for which an **overall EF and PO** allocation that is **PROP[1, 1] per round** exists.*