Avoiding Overrepresentation: Upper Quota Axioms for Committee Voting

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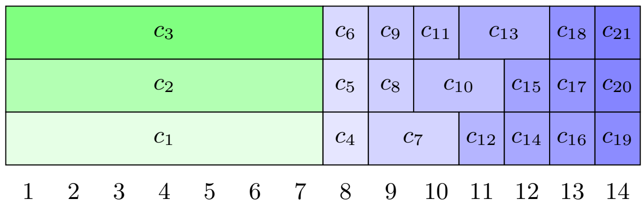
Abstract. Recently, in the social choice literature, much attention has been given to the question of avoiding underrepresentation in approval-based committee voting. In this paper, we explore the largely overlooked complementary question of avoiding overrepresentation. This has not been explored systematically, despite being a desirable property with concrete applications. Intuitively, overrepresentation happens when a group controls many more candidates in the committee than their size would warrant. We formulate a strong and appealing axiom for this, called *justifiable upper quota*. We prove that among the class of composite Thiele rules (a generalisation of the well-studied class of Thiele rules), there is a unique rule that satisfies this axiom. This rule is the natural analogue to a method proposed by John Quincy Adams in the context of parliamentary apportionment.

1 Introduction

Consider a scenario where a group of agents must jointly select a certain number of items from a pool of alternatives. Many real-world situations can be described in this way; for example, committee elections [10], parliamentary apportionment [2], or finding group recommendations [16].

This problem, known as *multi-winner voting* [8, 10], has recently been one of the most-studied issues in (computational) social choice [4]. Most of the effort was focused on *approval-based committee voting*, where voters express their preferences by *approving* or *disapproving* each candidate (see the survey by Lackner and Skowron [10]). In particular, one of the main issues in this area is how to guarantee *representation* or, equivalently, how to avoid underrepresentation. Briefly, the idea is that sizeable minorities deserve to be represented in the outcome: if a certain fraction of the voting population has similar preferences, they should approve at least roughly the same fraction of candidates in the committee. Much progress has been made in this direction, and many properties (or *axioms*) that formally capture this notion have been proposed (such as extended justified representation [1] or priceability [11]), as well as concrete voting rules that embody this property (for instance, PAV [9, 17], the method of equal shares [11], Phragmén's sequential rule [6]).

The main contribution of our paper is to look at the other side of the question, which has not received as much attention: *overrepresentation*. In short, this happens when a cohesive group of voters controls a fraction of candidates in the committee that is much larger than their size would warrant. To illustrate this, consider the following example. Suppose that a group of voters (say, half) is in complete agreement: they approve exactly the same candidates. None of the second half of the voters approve these candidates, and, additionally, their preferences are quite disjoint: no candidate is approved by more than a couple of voters. If the electorate is large enough, rules that avoid underrepresentation (such as PAV) here would give *all* the seats to the large group, even though this group constitutes only *half* of the population: the large group is overrepresented. Such a situation (for fourteen voters) is exemplified in the following picture:



Here, the horizontal axis corresponds to voters named from 1 to 14, and a voter approves all and only the candidates above her (for example, voter 11 approves of candidates c_{10}, c_{12} and c_{13}). Indeed, in this example, if we want to choose 3 candidates, PAV would select c_1, c_2 and c_3.

Depending on the application, avoiding overrepresentation can be as important as guaranteeing representation (or even more). In parliamentary apportionment [2, 12] (which is, formally, a special case of committee voting), avoiding overrepresentation is equivalent to what is known as *respecting upper quota* (or *upper frame*). This notion has received much attention and can be seen as a way to favour small states. Going in a more technical direction, Cevallos and Stewart [7] highlight the connection between overrepresentation and security in distributed systems, for example blockchain networks. More specifically, Boehmer et al. [3] study overrepresentation in the context of *Polkadot*, a blockchain with a mechanism based on multi-winner voting, insisting again on the link between overrepresentation and security. Note that both works focus on the quantitative study of overrepresentation, rather than its axiomatic analysis.

As mentioned earlier, despite the importance of this matter, previous work has mostly focused on avoiding underrepresentation. Indeed, rules that respect upper quota have received almost no attention. For example, while it is known that the aforementioned PAV rule is the only rule among the class of *Thiele rules* (a normatively

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attractive family of rules) that satisfies strong guarantees against underrepresentation (such as *d'Hondt proportionality* [9] or *extended* and *proportional justified representation* [1, 14]), no corresponding results are known for the issue of overrepresentation. Indeed, even defining a good upper quota axiom already poses some conceptual difficulties (which we explore in Section 3).

We intend to close this gap. Towards this end, we give two main contributions. First, we introduce a strong and appealing upper quota axiom, called *justifiable upper quota*. While this axiom is not satisfiable within the familiar class of Thiele rules, it is in the class of *composite Thiele rules*. The latter is analogous to the class of *composite social choice scoring functions* in standard single-winner voting [18]. Intuitively, composite Thiele rules are defined through a sequence of Thiele rules and work in stages. First, we compute the result of the first Thiele rule. Then, at each subsequent stage, we apply the next rule to break the ties of the previous stage. To the best of our knowledge, this is the first paper to introduce this class.

Our second main contribution (Theorem 2) shows that, among this general class, justifiable upper quota pinpoints a unique rule: *Adams-AV*. This rule is the natural extension to the committee voting setting of an apportionment method proposed by John Quincy Adams. The latter method is notable as the only population monotone[1] one that respects upper quota [2], which, as discussed above, corresponds to avoiding overrepresentation in the apportionment setting. To the best of our knowledge, although Lackner and Skowron [9] briefly mentioned this rule, our work is the first to initiate the axiomatic study of Adams-AV.

The paper is organised as follows. First, we introduce the necessary background (Section 2). In Section 3, we present some challenges in defining a proper upper quota axiom for committee voting, and introduce our main axiom: *justifiable upper quota*. We prove our main result (a characterisation of Adams-AV among composite Thiele rules in terms of JUQ) in Section 4. We discuss a strengthening of JUQ, which Adams-AV also satisfies, in Section 5. Section 6 concludes with pointers to future work.

2 Preliminaries

In this section, we introduce the model of *approval-based committee voting* [10] as well as a special case of it, *apportionment*. Then, we will introduce a family of voting rules called *composite Thiele rules*.

As usual, given a natural $\ell \in \mathbb{N}$, we define $[\ell] = \{1, \ldots, \ell\}$. Furthermore, given a set S, we denote by $\mathcal{P}(S)$ its powerset and as $\mathcal{P}_\ell(S) = \{T \in \mathcal{P}(S) \colon |T| = \ell\}$.

2.1 Approval-Based Committee Voting

Let $N = \{1, \ldots, n\}$ be the set of *voters* and C with $|C| = m > 0$ be the set of *candidates*. An *approval profile* (or simply *profile*) is a tuple $\boldsymbol{A} = (A_1, \ldots, A_n) \in \mathcal{P}(C)^n$ where $A_i \subseteq C$ for every $i \in [n]$. Given a candidate $c \in C$, we indicate its supporters as

$$N(c) = \{i \in N \colon c \in A_i\}$$

and the size of this set as $n_c = |N(c)|$.

Let $k \in \mathbb{N}$. We are interested in selecting a k-committee, that is, a k-sized subset of C. A *voting rule* is a function F that maps every profile $\boldsymbol{A}$ and committee size k to a nonempty $\mathcal{W} \subseteq \mathcal{P}_k(C)$. Ideally, we would want to select a single k-committee; we define F to be set-valued to model ties. Given two voting rules F and G, we say that F is a *refinement* of G if $F(\boldsymbol{A}, k) \subseteq G(\boldsymbol{A}, k)$ for all inputs.

Given a set of voters $N' \subseteq N$, we define its *quota* as

$$q(N') = k\frac{|N'|}{|N|}.$$

We also define $q_c = q(N(c))$. The quota of a group represents the fraction of candidates they deserve to have in the committee (assuming that the preferences of this group are similar enough). We call $\lceil q(N') \rceil$ the *upper quota* and $\lfloor q(N') \rfloor$ the *lower quota* of N'. Intuitively, if the group N' controls more than $\lceil q(N') \rceil$ candidates in the committee, it is *overrepresented*. We will define this notion formally later (Definition 4).

2.2 Apportionment

The *apportionment* setting is a special case of the committee voting model. A pair $\boldsymbol{A}$ and k is an *apportionment instance* if for every $i, j \in N$ we have either $A_i = A_j$ or $A_i \cap A_j = \emptyset$. Furthermore, for every $i \in N$ we must have $|A_i| = k$. An *apportionment method* is a voting rule whose domain is restricted to apportionment instances.

We introduce additional notation. We partition N into s disjoint equivalence classes called *parties*, that is, $N = P_1 \cup \cdots \cup P_t$. Two voters have the same ballot if and only if they belong to the same party. Given a committee W, we define the *seats* given to party P_ℓ as $a_\ell(W) = |A_i \cap W|$ (for some arbitrary $i \in P_\ell$). When W is fixed, we just write a_ℓ.

An apportionment method satisfies *apportionment upper quota* (resp., *lower quota*) if, for every outcome W selected by this method and nonempty party P_ℓ, we have that $a_\ell \leq \lceil q(P_\ell) \rceil$ (resp., $a_\ell \geq \lfloor q(P_\ell) \rfloor$).

2.3 Composite Thiele rules

A popular class of voting rules is the class of *Thiele rules*, introduced by Thorvald N. Thiele [17] in the late 19th century. These rules are well-studied in approval-based committee voting (see the book by Lackner and Skowron [10]). Intuitively, each Thiele rule is defined by a *weight function* which determines the score that every voter assigns to a committee given the number of candidates she approves in that committee.

In this paper, we consider a generalisation of this family, called *composite Thiele rules*. A composite Thiele rule is defined by a sequence of Thiele rules. First, we select a set of winning committees through the first rule. Then, we select a subset of the committees selected in the previous step through the second rule. This process continues until either the rules are finished or we have reached a fixed point.

Definition 1. *A* weight function *is a function $w \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $w(1) = 1$ and $w(s) \geq w(s+1)$ for every $s \in \mathbb{N}$. We sometimes write weight functions as vectors. Given $\boldsymbol{A}$, k, and $i \in N$, we define*

$$\mathrm{score}_w(A_i, W) = \sum_{s=1}^{|A_i \cap W|} w(s)$$

$$\textit{and} \quad \mathrm{score}_w(\boldsymbol{A}, W) = \sum_{j \in N} \mathrm{score}_w(A_j, W).$$

[1] Informally, population monotonicity refers to the idea that if a party receives more votes, it should not get fewer seats. The formal definition is given by Balinski and Young [2].

The simple *or* 1-composite Thiele rule *corresponding to* w *is then*

$$F_w(\boldsymbol{A}, k) = \underset{W \in \mathcal{P}_k(C)}{\arg\max} \;\; \mathrm{score}_w(\boldsymbol{A}, W).$$

Given two weight functions w_1 *and* w_2*, define*

$$(F_{w_2} \circ F_{w_1})(\boldsymbol{A}, k) = \underset{W \in F_{w_1}(\boldsymbol{A},k)}{\arg\max} \;\; \mathrm{score}_{w_2}(\boldsymbol{A}, W).$$

Given any sequence $\boldsymbol{w} = (w_1, w_2, \ldots)$ *of length* $\ell \in \mathbb{N} \cup \{\infty\}$ *of pairwise-distinct weight functions, the corresponding* ℓ-composite Thiele rule *is*

$$F_{\boldsymbol{w}}(\boldsymbol{A}, k) = \bigcap_{i=1}^{\ell} (F_{w_i} \circ \cdots \circ F_{w_1})(\boldsymbol{A}, k).$$

Note that, if $\boldsymbol{w} = (w_1, \ldots, w_\ell)$ *is finite, then the above collapses to*

$$F_{\boldsymbol{w}}(\boldsymbol{A}, k) = (F_{w_\ell} \circ \cdots \circ F_{w_1})(\boldsymbol{A}, k).$$

We allow for ∞-composite Thiele rules to also capture rules such as the one given by the infinite sequence $(1, 0, 0, \ldots)$, $(1, 1, 0, 0, \ldots)$, $(1, 1, 1, 0, 0, \ldots)$, etc. This is an analogue to the leximin social welfare function [13, 15].[2]

To illustrate our definition further, we present some standard Thiele voting rules.

Definition 2. *We define three simple Thiele voting rules:*

- Approval Voting *(AV) is the Thiele rule given by* $w = (1, 1, 1, \ldots)$.
- Chamberlin-Courant *(CC) is the Thiele rule given by* $w = (1, 0, 0, \ldots)$.
- Proportional AV *(PAV) is the Thiele rule given by* $w = (1, 1/2, 1/3, \ldots)$.

Let us give an intuitive idea of these rules. AV simply picks the k-committees with the highest number of approved candidates. On the other hand, CC selects the k-committees that cover as many voters as possible, where a voter is covered if she approves at least one candidate in the committee. One can think of AV and CC as lying on two different ends of the spectrum of (simple) Thile rules: while AV is entirely utilitarian, CC tries to satisfy all voters whenever possible. PAV lies, in some sense, in between these two. Given a k-committee W, under PAV each voter assigns to W a score corresponding to the $|A_i \cap W|$-th harmonic number; the k-committees with the highest total score are selected. The intuition here is that any additional candidate a voter approves in the committee will yield diminishing returns.

Restricted to the apportionment setting, PAV coincides with *d'Hondt's method* (also known as *Jefferson's*), which is notable as the only apportionment method that satisfies lower quota as well as population monotonicity [2]. In fact, PAV is the only Thiele rule consistent with d'Hondt [9], as well as the only one that satisfies PJR [14] and EJR [1], two popular lower quota axioms in the committee-voting setting. In light of this, we focus on the following rule.

Definition 3 (Adams-AV)**.** Adams-AV *is the* 2*-composite Thiele rule given by* $w_1 = (1, 0, 0, \ldots)$ *and* $w_2 = (1, 1, 1/2, 1/3, \ldots)$.

[2] Observe that, since we assume n to be fixed, we do not technically need infinite sequences to capture this rule. However, we stick with this definition (which accommodates for electorates of variable size) as our results also hold for this more general case.

Intuitively, Adams-AV can be seen as a mixture between CC and (a variant of) PAV. First, we select the committees that cover as many voters as possible. Among those, we select the committees that maximise the following score: each voter assigns to every W a score corresponding to the $(|A_i \cap W| - 1)$-th harmonic number (or 0 if $|A_i \cap W| = 0$).

Example 1. Consider the following profile: $\boldsymbol{A} = (\{a, b, c, d\}, \{a, b, c, d\}, \{a, b, c, e\}, \{d\}, \{b, c, e\}, \{c, e\})$ and let $k = 2$. Here, AV picks only $\{b, c\}$, the two most approved candidates. CC selects $\{c, d\}$ and $\{d, e\}$, as they both cover all voters. PAV selects $\{b, c\}$ and $\{c, d\}$. Finally, among the two CC-optimal committees, Adams-AV picks only committee $\{c, d\}$. $\triangle$

Adams-AV, when restricted to the apportionment setting, coincides with a method proposed by John Quincy Adams, which is the only apportionment method that respects upper quota and satisfies population monotonicity [2]. It is thus natural to wonder, mirroring the above discussion about PAV, whether Adams-AV can be characterised among composite Thiele rules by an appropriate upper quota axiom. Indeed, as we shall see (Theorem 2), this is the case.

3 Justifiable Upper Quota

In this section, we will define our main upper quota axiom.

We will start with a discussion on why defining upper quota axioms in the context of committee voting poses some additional challenges compared to lower quota.

Example 2. Consider profile $\boldsymbol{A} = (\{a, b, c\}, \{d, e, f\}, \{g\})$ and let $k = 6$. This is *almost* an apportionment instance, but here voters approve fewer than k candidates. Since no two voters jointly approve of any candidate, one would expect here to have three groups of voters (each consisting of a single voter) that must not be overrepresented. However, all three singleton groups have an upper quota of $\lceil 6 \cdot 1/3 \rceil = 2$ and, for any k-committee W, at least one voter is approving 3 candidates in W. Hence, every outcome overrepresents at least one group. $\triangle$

This example suggests that any satisfiable upper quota axiom should either (1) *not* consider the first two voters as two distinct groups subject to upper quota, or (2) have additional conditions that allow for some overrepresentation when unavoidable.

We believe that option (1) would require a rather unnatural definition of what a "group" of voters is, and that option (2) results in a much more intuitively appealing axiom. We propose the following.

Definition 4 (Justifiable Upper Quota)**.** *Committee* W *satisfies* Justifiable Upper Quota *(JUQ) if, whenever there is a* $c \in W$ *such that* $N(c) \neq \emptyset$ *and*

$$|A_i \cap W| > \lceil q_c \rceil \quad \text{for all } i \in N(c),$$

then there exists no $d \in C \setminus W$ *such that* $N(d) \neq \emptyset$ *and*

$$|A_i \cap (W \cup \{d\} \setminus \{c\})| \leq \lceil q_d \rceil \quad \text{for all } i \in N(d).$$

The intuition behind this axiom is as follows. We consider a committee that overrepresents a group to violate our axiom only if there is another group which could receive an additional candidate and still not be overrepresented. Indeed, the term *justifiable* stems from this:

we should be able to justify each violation of upper quota by arguing that any swap from the current committee would still entail a violation.

Even when every outcome overrepresents some group, JUQ is still satisfiable.

Example 3. Let us revisit the scenario of Example 2, where $\boldsymbol{A} = (\{a, b, c\}, \{d, e, f\}, \{g\})$ and $k = 6$. Consider, e.g., outcome $W_1 = \{a, b, c, d, e, g\}$. This satisfies JUQ. Indeed, consider, for example, $c \in W_1$. We have that $\lceil q_c \rceil = 2$, but voter 1 has satisfaction 3. However, the only candidate not in W_1 is f, and we have that, if we were to replace c with f in W_1, then voter 2 would get a satisfaction of $3 > 2 = \lceil q_f \rceil$. On the other hand, outcome $W_2 = \{a, b, c, d, e, f\}$ violates JUQ. Indeed, again c is a witness for an upper quota violation. However, now we should replace c with g, as $\lceil q_g \rceil = 2 > 1$. $\triangle$

We now discuss our axiom further. Observe that we require the following: if *all* members in a group $N(c)$ are above their upper quota, and there is another group $N(d)$ where *all* members would be below upper quota after replacing c with d, then we should do so. Note the asymmetry: it would be perhaps more natural to require only for *some* member of $N(d)$ to be below her upper quota to warrant the replacement of c by d. Our weaker notion is already enough to characterise Adams-AV among composite Thiele rules, and thus we believe it is strong enough to be useful. However, we note that Adams-AV would not satisfy the alternative axiom outlined above. Indeed, consider 8-voter profile

$$\boldsymbol{A} = (\{a\}, \{a\}, \{a, d\}, \{a, b, c\}, \{a, b, c\}, \{a, b, c\}, \\ \{a, b, c, d\}, \{a, b, c, d\})$$

and let $k = 3$. Adams-AV here picks only $\{a, b, c\}$. However, $N(c)$ is overrepresented as all its members have satisfaction $3 > 2 = \lceil q_c \rceil$. If we switch c with d, then $N(d)$ would not be overrepresented (under the alternative definition) as the voter with ballot $\{a, d\}$ would have satisfaction $2 = \lceil q_d \rceil$. Hence, Adams-AV violates this variant of JUQ.

Furthermore, we implicitly assume that the only groups of voters that are not to be overrepresented are those of the form $N(c)$ for some candidate $c \in C$. This is similar to the notion of a "group" that underpins well-studied lower quota axioms such as EJR+ [5], but slightly more restrictive. Indeed, one could relax our notion by also requiring that any N' with $N' \subseteq N(c)$ for some $c \in C$ should not be overrepresented. We discuss the resulting axiom in Section 5 and show that Adams-AV still satisfies this. Here, we stick with the current definition as it results in a weaker axiom and thus in a stronger characterisation.

Moreover, note that we require both $N(c)$ and $N(d)$ to be nonempty. The two conditions, together, make JUQ more agnostic to efficiency considerations; the first one avoids *excluding* inefficient solutions, while the second one avoids *restricting attention to* inefficient solutions. Indeed, suppose $C = \{a, b\}$, $\boldsymbol{A} = (\{a\}, \{a\})$ and $k = 1$. If we did not require $N(c) \neq \emptyset$, then $W = \{b\}$ would violate JUQ. However, although this committee is clearly inefficient, it does not overrepresent any group. In general, if we did not require $N(c)$ to be nonempty, then JUQ would enforce the exclusion of every candidate c with no supporters, unless replacing this candidate with some other d would overrepresent $N(d)$. On the other hand, consider $C = \{a, b, c, d, e\}$, $\boldsymbol{A} = (\{a\}, \{b, c, e\})$ and $k = 4$. If we did not require $N(d) \neq \emptyset$, then $\{a, b, c, e\}$ would violate JUQ, as the latter would require replacing c with d. This is, however, clearly an inefficient solution. With the current Definition 4, $\{a, b, c, e\}$ satisfies JUQ (as do other committees, such as $\{a, b, c, d\}$).

Finally, a natural requirement for upper quota axioms is to extend apportionment upper quota, which is indeed the case for JUQ.

Proposition 1. *For apportionment instances, JUQ is equivalent to apportionment upper quota.*

Proof. Consider an apportionment instance $\boldsymbol{A}$ and k. Fix a k-committee W. If W satisfies apportionment upper quota, then it trivially satisfies JUQ. Then, suppose it satisfies JUQ.

Towards a contradiction, assume that there is some $c \in W$ with $N(c) \neq \emptyset$ and $|A_i \cap W| > \lceil q_c \rceil$ for all $i \in N(c)$. Without loss of generality, assume $N(c) = P_1$. By JUQ we must have, for all $d \in C \setminus W$ with nonempty $N(d)$, that $|A_i \cap (W \cup \{d\} \setminus \{c\})| > \lceil q_d \rceil$ for some (equivalently, all) $i \in N(d)$. This implies that every party P_ℓ has $a_\ell \geq \lceil q(P_\ell) \rceil$ and that $a_1 \geq \lceil q(P_1) \rceil + 1$. Hence,

$$k \geq \sum_{\ell=1}^{t} a_\ell > \sum_{\ell=1}^{t} \left\lceil k \cdot \frac{|P_\ell|}{n} \right\rceil \geq \left\lceil \sum_{\ell=1}^{t} k \cdot \frac{|P_\ell|}{n} \right\rceil = k,$$

where t is the number of parties. This is a contradiction: hence, for no $c \in W$ with $N(c) \neq \emptyset$ we have $|A_i \cap W| > \lceil q_c \rceil$ for all $i \in N(c)$. This means that W satisfies apportionment upper quota. $\square$

4 Main result: JUQ characterises Adams-AV

We now state and prove our main result: JUQ characterises Adams-AV in the class of composite Thiele rules.

Theorem 2. *A composite Thiele rule satisfies justifiable upper quota if and only if it is a refinement of Adams-AV.*

Note that the second part of the statement (only Adams-AV can satisfy JUQ) can already be derived from the theory of apportionment, at least when it comes to simple Thiele rules. Indeed, we know that every simple Thiele rule corresponds to a unique *divisor method* (a class of normatively appealing apportionment methods) when restricted to apportionment instances [9]. Since all divisor methods except Adams' violate apportionment upper quota [2], we know from Proposition 1 that no simple Thiele rule can satisfy JUQ. Nonetheless, we give a direct proof of our claim.

Proof of Theorem 2. First, we show that any refinement F of Adams-AV satisfies JUQ. Let $w_1 = (1, 0, 0, \ldots)$ and $w_2 = (1, 1, 1/2, 1/3, \ldots)$. Fix a profile $\boldsymbol{A}$ and committee size k. Let W be a k-committee selected by F and suppose towards a contradiction that it violates JUQ, i.e., we could swap some $c \in W$ with some $d \in C \setminus W$.

In the following, let $W' = (W \setminus \{c\}) \cup \{d\}$. Since $n_c > 0$, $\lceil q_c \rceil \geq 1$ and consequently $|A_i \cap W| \geq 2$ for every $i \in N(c)$. With this, we derive that all i with $|A_i| > 0$ are covered by both W (as otherwise we could remove c from W and cover an additional voter) and W' (as only the members of $N(c)$ can suffer a decrease in satisfaction from W to W'). Thus, given the Adams-AV-optimality of W, we must have $\text{score}_{w_2}(\boldsymbol{A}, W) \geq \text{score}_{w_2}(\boldsymbol{A}, W')$. Since we only swap c and d, the satisfaction of every $i \in N$ with $\{c, d\} \subseteq A_i$ or $\{c, d\} \cap A_i = \emptyset$ is the same for both W and W'. Every other voter has satisfaction of at least 1, and hence we must have

$$\sum_{i \in N(c)\setminus N(d)} \sum_{s=2}^{|A_i \cap W|} {}^{1}\!/_{(s-1)} + \sum_{i \in N(d)\setminus N(c)} \sum_{s=2}^{|A_i \cap W|} {}^{1}\!/_{(s-1)} \geq \sum_{i \in N(c)\setminus N(d)} \sum_{s=2}^{|A_i \cap W'|} {}^{1}\!/_{(s-1)} + \sum_{i \in N(d)\setminus N(c)} \sum_{s=2}^{|A_i \cap W'|} {}^{1}\!/_{(s-1)},$$

which, given that $|A_i \cap W'| = |A_i \cap W| - 1$ for $i \in N(c) \setminus N(d)$ and $|A_i \cap W'| = |A_i \cap W| + 1$ for $i \in N(d) \setminus N(c)$, yields

$$\sum_{i \in N(c)\setminus N(d)} \frac{1}{|A_i \cap W| - 1} \geq \sum_{i \in N(d)\setminus N(c)} \frac{1}{|A_i \cap W|}.$$

By definition of JUQ, the satisfaction of any $i \in N(c)$ is at least $\lceil q_c \rceil + 1$, while the satisfaction of any $i \in N(d)$ is at most $\lceil q_d \rceil - 1$. Hence, the above inequality implies ${}^{(n_c-\ell)}\!/_{\lceil q_c \rceil} \geq {}^{(n_d-\ell)}\!/_{(\lceil q_d \rceil - 1)}$ with $\ell = |N(c) \cap N(d)|$, and thus

$$\frac{n_c - \ell}{k \cdot {}^{n_c}\!/_n} \geq \frac{n_c - \ell}{\lceil k \cdot {}^{n_c}\!/_n \rceil} \geq \frac{n_d - \ell}{\lceil k \cdot {}^{n_d}\!/_n \rceil - 1} > \frac{n_d - \ell}{k \cdot {}^{n_d}\!/_n}.$$

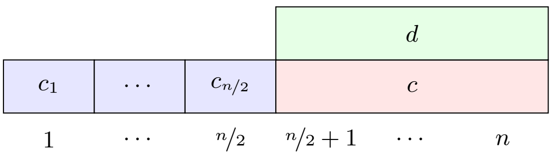
Therefore, $\ell > 0$ and $n_c > n_d$ must hold. These two facts imply, respectively, that there is some $i^\star \in N(c) \cap N(d)$ and that $\lceil q_c \rceil \geq \lceil q_d \rceil$. Together with the definition of JUQ, this means that $\lceil q_c \rceil \geq \lceil q_d \rceil \geq |A(i^\star) \cap W'| = |A(i^\star) \cap W| > \lceil q_c \rceil$, our desired contradiction.

Next, we show that any other composite Thiele rule fails JUQ. Fix $\boldsymbol{w}$ and suppose $F_{\boldsymbol{w}}$ satisfies JUQ. We will work in three steps, using three profiles.

1. First, we show that $w_1 = (1, 0, \ldots)$.
2. Second, we show that $\boldsymbol{w}$ has at least two entries and that $w_2(s) > 0$ for all $s \in \mathbb{N}$.
3. Finally, we show that $w_2 = (1, 1, {}^1\!/_2, {}^1\!/_3, \ldots)$.

The above is sufficient to show that $F_{\boldsymbol{w}}$ refines Adams-AV.

First, fix $k = 2$ and consider $n \in 2\mathbb{N}$ with $n > 2$. Suppose that there are ${}^n\!/_2 + 2$ candidates, namely, $C = \{c_1, \ldots, c_{n/2}, c, d\}$. Construct the following profile:

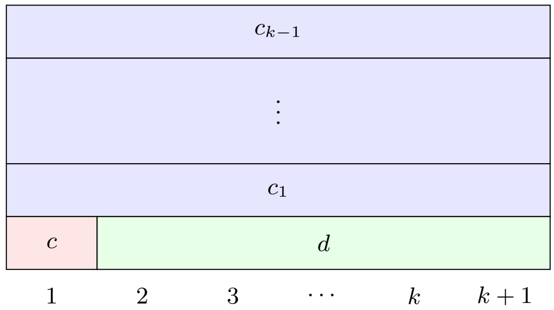


In words, every voter $i \in [{}^n\!/_2]$ approves $\{c_i\}$, while the remaining ${}^n\!/_2$ voters all approve $\{c, d\}$. Since $w_1(1) = 1$, any 2-committee W with $W \cap \{c, d\} = \emptyset$ cannot be in the outcome, as we would increase $\text{score}_{w_1}(\boldsymbol{A}, k)$ by substituting any $c_i \in W$ with either c or d. Moreover, $\lceil q_x \rceil = 1$ for all $x \in C$, and hence $W = \{c, d\}$ must lose to any 2-committee W' containing exactly one candidate from $\{c, d\}$. Therefore,

$${}^n\!/_2 \cdot (w_1(1) + w_1(2)) \leq ({}^n\!/_2 + 1) \cdot w_1(1) \implies {}^n\!/_2 \cdot w_1(2) \leq w_1(1).$$

Since w does not depend on n and n can be arbitrarily large, $w_1(2) = 0$.

Next, fix $k > 2$. We construct a profile with $k+1$ voters and $k+1$ candidates as follows:

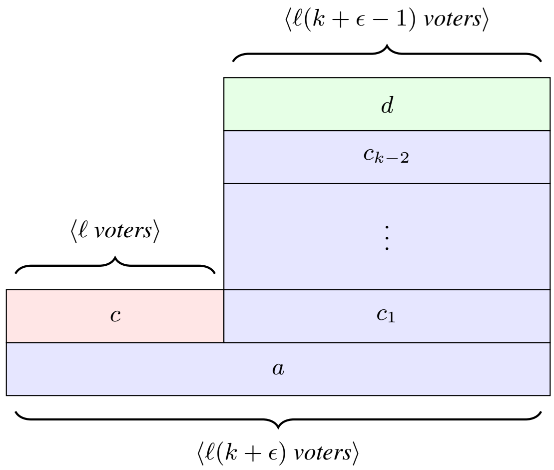


In words, let C be partitioned as $C = C' \cup \{c\} \cup \{d\}$. Voter 1 approves $C' \cup \{c\}$, whereas every voter in $N \setminus \{1\}$ approves $C' \cup \{d\}$. Observe that the k-committee $W = C' \cup \{c\}$ would violate JUQ, as $\lceil q_c \rceil = \lceil {}^k\!/_{(k+1)} \rceil = 1 < k = |A(k+1) \cap W|$ and $\lceil q_d \rceil = \lceil k \cdot {}^{k^2}\!/_{(k+1)} \rceil = k$. This shows that $\boldsymbol{w}$ must contain at least two entries; since every committee covers all voters, all k-committees are w_1 optimal and thus W would be among the selected outcomes. Hence, assume $\boldsymbol{w} = (w_1, w_2, \ldots)$ with $w_2(2) > 0$ (recall that weight functions must be pairwise-distinct). This fact implies, in turn, that any winning committee cannot contain both c and d, as otherwise we could increase the w_2-score of the outcome. Hence, W must lose to $(W \setminus \{c\}) \cup \{d\}$, and therefore

$$\sum_{s=1}^{k} w_2(s) + k \sum_{s=1}^{k-1} w_2(s) < \sum_{s=1}^{k-1} w_2(s) + k \sum_{s=1}^{k} w_2(s) \implies w_2(k) < k \cdot w_2(k).$$

Since k was arbitrarily chosen, we have proven that w_2 is strictly positive on all inputs.

From now on, we can safely assume that $w_2(1) = w_2(2) = 1$. Fix $k > 2$. Pick an $\epsilon \in \mathbb{Q} \setminus \{0\}$ where $|\epsilon|$ is arbitrarily small and some $\ell \in \mathbb{N}$ such that $|\ell \cdot \epsilon| \in \mathbb{N}$. Let there be $n = \ell(k + \epsilon)$ voters and $k + 1$ candidates. There are three distinguished candidates, namely, a, c and d. Construct the following profile:



In words, we have a group of $\ell(k + \epsilon - 1)$ voters who approve all candidates in $C \setminus \{c\}$, including a and d. The remaining ℓ voters only approve $\{a, c\}$. First, any k-committee covers all voters and since $w_2(s) > 0$ for all $s \in \mathbb{N}$, we cannot pick committee $C \setminus \{a\}$ as we could replace d with a and strictly improve the w_2-score. Next,

$$q_c = k \cdot \frac{\ell}{\ell(k + \epsilon)} \quad \text{and} \quad q_d = k \cdot \frac{\ell(k - 1 + \epsilon)}{\ell(k + \epsilon)}.$$

We distinguish two cases based on the sign of ϵ.

- *Assume* $\epsilon > 0$. Then, one can compute $\lceil q_c \rceil = 1$ and $\lceil q_d \rceil = k$. Consequently, we must not pick any W with $c \in W$, as otherwise we would have $|A_i \cap W| = 2 > \lceil q_c \rceil$ for all $i \in N(c)$ while the other group is all strictly under its upper quota. Therefore, given any committee W' that does not contain c (but contains a),

$$\ell(w_2(1) + w_2(2)) + \ell(k + \epsilon - 1) \sum_{s=1}^{k-1} w_2(s) < \ell w_2(1) + \ell(k + \epsilon - 1) \sum_{s=1}^{k} w_2(s),$$

which implies

$$\frac{w_2(2)}{k + \epsilon - 1} < w_2(k).$$

Since $w_2(2) = 1$ and since this holds for any $k \geq 3$ and arbitrarily small $\epsilon > 0$ we get that $w_2(s) \geq 1/(s-1)$ for all $s > 1$.
- *Assume* $\epsilon < 0$. Then, one can compute $\lceil q_c \rceil \geq 2$ and $\lceil q_d \rceil < k$. In contrast to the previous case, we must not pick $W = C \setminus \{c\}$. By analogous computations we obtain $w_2(k) < \frac{w_2(2)}{k+\epsilon-1}$ and thus $w_2(s) \leq 1/s-1$ for all $s > 1$.

Combining the two cases, we have shown $w_2(s) = 1/s-1$ for $s > 1$, completing the proof. ☐

5 Strengthening JUQ

We conclude by discussing a strengthening of JUQ.

As noted in Section 3, our definition of JUQ focuses on groups of voters $N(c)$ for some $c \in C$. One could also consider subsets of such sets. This is similar to what is done, for instance, in EJR+ [5], an appealing lower quota axiom. Towards this end, we introduce the following axiom.

Definition 5 (Subset Justifiable Upper Quota)**.** *Committee W satisfies* Subset Justifiable Upper Quota *(SJUQ) if, whenever there is a $c \in W$ such that $N(c) \neq \emptyset$ and*

$$|A_i \cap W| > \lceil q_c \rceil \quad \textit{for all } i \in N(c),$$

then there exists no $d \in C \setminus W$ and nonempty $N' \subseteq N(d)$ such that

$$|A_i \cap (W \cup \{d\} \setminus \{c\})| \leq \lceil q(N') \rceil \quad \textit{for all } i \in N'.$$

Note that one could go further in this direction and apply the same idea to $N(c)$. Indeed, in the first part of the definition, we could have that "whenever there is a $c \in W$ and a nonempty $N^\dagger \subseteq N(c)$ such that $|A_i \cap W| > \lceil q(N^\dagger) \rceil$ for all $i \in N^\dagger$ [...]". This would not be an appealing definition: it would imply, for instance, that whenever $k \leq n$, all voters must approve at most 1 candidate in the committee.

SJUQ implies JUQ by definition, but the converse is not true.

Example 4. Let $\boldsymbol{A} = (\{a\}, \{d\}, \{a, b, c, d\})$ and $k = 4$. Committee $W = \{a, b, c\}$ satisfies JUQ, but not SJUQ. Indeed, we have that $N(c)$ consists only of voter 3, who is approving 3 candidates in W while $\lceil q_c \rceil = 2$. However, the only candidate outside W is d, and voter 3 is in $N(d)$. Hence, JUQ is satisfied.

On the other hand, SJUQ is not satisfied. Indeed, consider the subset $N' \subseteq N(d)$ consisting only of voter 2. Then, if we replace c with d in the outcome, we would have that this voter approves 1 candidate in the committee, and that $\lceil q(N') \rceil = 2$. Hence, SJUQ is violated.

Observe that, here, $W = \{a, b, d\}$ satisfies both axioms. △

We get the following.

Theorem 3. *Any refinement of Adams-AV satisfies SJUQ.*

Proof. We assume $k > 1$, as otherwise the proof is trivial. Suppose that W is selected by some refinement of Adams-AV. Assume towards a contradiction that there is some c, d, and N' as described in the definition of SJUQ (Definition 5). Let $|N'| = n'$ and $|N(d) \setminus N'| = n^\dagger$. Moreover, let $\ell' = |N(c) \cap N'|$ and $\ell^\dagger = |N(c) \cap (N(d) \setminus N')|$. Finally, let $q' = \lceil k \cdot n'/n \rceil$. By noticing that all voters in $N(d) \setminus N' \setminus N(c)$ have satisfaction at most $k-1$ with W, with arguments analogous to the first part of the proof of Theorem 2, we get

$$\frac{(n_c - \ell' - \ell^\dagger)}{\lceil q_c \rceil} \geq \frac{(n' - \ell')}{(\lceil q' \rceil - 1)} + \frac{(n^\dagger - \ell^\dagger)}{k-1} \implies$$
$$\frac{n(n_c - \ell' - \ell^\dagger)}{k \cdot n_c} > \frac{n(n' - \ell')}{k \cdot n'} + \frac{(n^\dagger - \ell^\dagger)}{k-1} \implies$$
$$\ell' \left(\frac{1}{n'} - \frac{1}{n_c} \right) > \frac{k}{n(k-1)} \left(n^\dagger - \ell^\dagger \right) + \frac{\ell^\dagger}{n_c}.$$

Since $n^\dagger \geq \ell^\dagger$, we derive that $\ell' > 0$ and $n_c \geq n'$. The proof concludes like the first part of the proof of Theorem 2. ☐

6 Conclusion and research directions

In this paper, we have defined an upper quota axiom for approval-based committee elections, called justifiable upper quota (Definition 4). Defining this property was non-trivial and, to the best of our knowledge, this is the first axiom of this kind in the literature. Our notion is interesting in that it characterises Adams-AV among the class of composite Thiele rules (Theorem 2). Given that in the apportionment setting Adams' is the only divisor method satisfying upper quota, and given the connection between Thiele rules and divisor methods [9], characterising Adams-AV among composite Thiele rules seems a natural consistency-check for any reasonable upper quota axiom. Furthermore, we have considered a strengthening of our axiom (Definition 5), and shown that Adams-AV also satisfies this (Theorem 3).

We suggest different avenues for future research. Firstly, one could extend the study of upper quota axioms beyond composite Thiele rules. Consider Phragmén's sequential rule (see the paper by Brill et al. [6] and the references therein) or the method of equal shares [11] with standard completion methods (i.e., completion by budget increase, by AV, or by sequential Phragmén). All these rules satisfy interesting lower quota notions [10]. However, all fail JUQ. For example, if $\boldsymbol{A} = (\{a\}, \{a\}, \{b\}, \{a, c\}, \{a, c\})$ and $k = 2$, sequential Phragmén and the method of equal shares with standard completion methods all pick $\{a, c\}$, which fails JUQ. Moreover, with profile

$$\boldsymbol{A} = (\{a, e\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, d, e\}, \{a, b, d, e\}, \{a, b, d, e\}, \{a, c, e, f\}, \{a, b, d, e, f\})$$

and $k = 5$, the method of equal shares without completion returns (among others) $\{a, b, c, e\}$, which fails JUQ. Is there any (possibly non-exhaustive) variant of these rules that satisfies JUQ? If not, is there any other poly-time computable rule that satisfies it?

More generally, it is important to understand the relationship between JUQ and standard lower quota axioms such as EJR or priceability, possibly without requiring exhaustiveness (i.e., returning exactly k candidates). Are they compatible? If so, is there any natural rule that satisfies both? Based on our previous discussion, we know that many attractive rules that satisfy such axioms fail JUQ. On the other hand, we know that JUQ is compatible at least with JR, a weak lower quota axiom, as JR is satisfied by CC [1] (and thus by Adams-AV).

Next, one could look at other quota notions. For example, the *Sainte-Laguë* (also known as *Webster*'s) apportionment method is the only divisor method that stays *near quota* [2]. That is, it is never possible to transfer a seat from one party to another while bringing both parties closer to their quotas. Can one define such an axiom in the context of committee voting, and characterise the corresponding committee voting rule (known as SLAV [10]) through this notion?

Finally, one could extend our partial characterisation of Adams-AV to a full characterisation. Indeed, we have used JUQ to characterise this rule among composite Thiele rules. What axioms are necessary to fully characterise it? The axiomatic characterisation of committee scoring rules proposed by Lackner and Skowron [9] is a natural starting point. It would be interesting to see whether by relaxing *continuity* (one of the axioms they use) one could achieve a similar characterisation of *composite* scoring rules and thus of Adams-AV. Indeed, in the context of single-winner voting, continuity acts as a separator between composite and non-composite social choice scoring functions in the analogous characterisation by Young [18].

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