Free-Riding in Multi-Issue Decisions

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Voting in multi-issue domains allows for compromise outcomes that satisfy all voters to some extent. Such fairness considerations, however, open the possibility of a special form of manipulation: free-riding. By untruthfully opposing a popular opinion in one issue, voters can receive increased consideration in other issues. We study under which conditions this is possible and show that even weak fairness considerations enable free-riding. Additionally, we study free-riding from a computational and experimental point of view. Our results show that free-riding in multi-issue domains is often possible, but comes at a non-negligible individual risk for voters. Thus, the allure of free-riding is smaller than one could intuitively assume.

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1 Introduction

In our increasingly digital and interconnected world, many new collective decision problems arise, with applications ranging from social media to the fine-tuning of AI systems through human feedback (Alamdari et al. 2024; Bakker et al. 2022). In contrast to traditional collective decisions like political elections, these new domains often require much more complex and interlinked decisions, e.g., by requiring the same voter (evaluator) to make many different decisions about the desired behavior of an AI system. In practice, such decisions are often made using simple majoritarian aggregation rules, which leads to underrepresentation of minority opinions and bias.[1] Recent research in the area of Computational Social Choice has shown that we can often leverage the fact that several different issues need to be decided to achieve a fairer comprise outcome (Chandak et al. 2024; Lackner 2020; Lackner and Maly 2023; Lackner and Skowron 2023; D. Peters 2024). The combinatorial richness of a sequence of decisions opens the opportunity for incorporating a more diverse set of opinions, and to respect a broader range of preferences.

However, by striving for fairness across multiple issues, we open the door to a specific, simple form of manipulation: *free-riding*. We define free-riding as untruthfully opposing a necessarily winning candidate. That is, if there is a very popular (maybe unanimous) alternative for a certain issue, it typically does not change the outcome if one voter does not approve of it. Under the assumption that multi-issue voting rules try to

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[1]This issue is discussed in the context of reinforcement learning from human feedback, e.g., in a recent survey (Casper et al. 2023, in Section 3.2.1), in the context of participatory budgeting (Fairstein et al. 2023), and the aggregation of moral judgments (Chandak et al. 2024).

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establish fairness based on satisfaction, it will consequently give this voter additional consideration in other issues. Consequently, presented with a popular alternative that is certain to win, a voter may be tempted to misrepresent their preferences and by that artificially lowers their (calculated) satisfaction. As we show in our paper, this form of manipulation is possible almost universally in multi-issue voting.

Thus, it may appear that free-riding is a form of simple and risk-free manipulation. The main contribution of our paper is to challenge this intuition. While free-riding is indeed often a successful form of manipulation, it is far from trivially beneficial for free-riding individuals. For our analysis, we focus on multi-issue decision making scenarios in which there are two or more alternatives per issue and voters submit one approval ballot per issue. We are mostly interested in the setting where issues are decided sequentially, because free-riding occurs very naturally in this setting. However, we also consider the case where issues are decided simultaneously, i.e., where voters present their preferences for all issues at the same time, and all issues are decided in one shot. Within both scenarios, we consider voting rules based on order-weighted averages, short: OWA (Amanatidis et al. 2015; Yager 1988), and on Thiele score, inspired by multi-winner voting (Lackner and Skowron 2023; Thiele 1895). We obtain the following results:

- First, we show via axiomatic analysis that essentially every reasonable voting rule is susceptible to free-riding, both in the sequential and the non-sequential case. The only notable exception is the utilitarian rule, which maximizes the sum of utilities and which is thus a fully majoritarian rule.
- For sequential OWA and Thiele rules, we observe an interesting phenomenon. Here, it may be that free-riding in an issue leads to a *lower* satisfaction in subsequent issues. Thus, free-riding for these voting rules is not risk-free.
- Moreover, we show that it is a computationally hard task to determine whether free-riding is beneficial in the long run. This is in contrast to simply determining whether free-riding is possible. Note that the computation of long-term consequences requires full preference information about all (subsequent) issues; the decision to free-ride requires only information about a single issue. Hence, free-riding is either inherently risky (in the case of incomplete information) or determining its eventual consequences is at least computationally difficult (in the case of complete information).
- Voters may still decide to free-ride if the risk is small enough. To study this question, we complement our theoretical analysis with numerical simulations to quantify this risk. In case that a voter can free-ride only in one issue (e.g., because it is difficult to recognize popular alternatives), our simulations show that the risk of free-riding is indeed significant. Nonetheless, positive outcomes of free-riding are more likely than negative outcomes; the exact risk depends largely on the voting rule in use. In case that a voter can repeatedly free-ride (i.e., using every free-riding opportunity), the risk of free-riding becomes negligible. This is because positive outcomes of free-riding are more likely, and hence repeated free-riding makes net-negative outcomes very unlikely.
- Finally, we consider the non-sequential setting. We study global optimization rules, which decide all issues at once. We show that, even when the winner of an issue is already known, it remains computationally hard to determine whether free-riding is possible (without changing the winner). Thus, in this setting, even single-issue free-riding is computationally difficult and requires full information. Hence, we conclude that for global optimization rules, the problem of free-riding is essentially the same as the more general problem of strategic voting, since both require full information and high computational power.

In general, we conclude that free-riding in multi-issue voting is not as simple and risk-free as one could intuitively assume. Our results paint a nuanced picture of where free-riding can easily occur and where it is difficult to free-ride.

1.1 Structure

Our paper is structured as follows. In Section 2, we introduce our model for multi-issue elections and free-riding. This is followed by our theoretical contribution related to the possibility and risk of free-riding (Section 3) and regarding the computational complexity of free-riding (Section 4). We present numerical simulations in Section 5. While the majority of this paper is based on sequential voting rules (deciding one issue after the other), we consider the viewpoint of global optimization rules in Section 6. Finally, in Section 7, we summarize our results and discuss directions for future work. We have moved especially long or technical proofs to Appendix B (computational complexity) and C (global optimization rules).

This work is an extended version of a previous conference publication (Lackner, Maly, and Nardi 2023).

1.2 Related Work

Our work falls in the broad domain of *voting in combinatorial domains* (Lang and Xia 2016). In contrast to many works in this field (e.g., Ahn and Oliveros 2012; Brams et al. 1998, 1997; Conitzer and Xia 2012; Lang and Xia 2009)), we assume that voters' preferences are separable (i.e., independent) between issues. Separability is a very natural assumption in our sequential setting, as voters may be only asked to state their preferences after preceding issues have already been decided.

Our work is most closely related to papers on multiple referenda. In this setting, Amanatidis et al. (2015) study the computational complexity of OWA voting rules, including questions of strategic voting. In a similar model, Barrot et al. (2017) consider questions of manipulability: how does the OWA vector impact the susceptibility to manipulation? In contrast to our paper, these two papers do not consider free-riding. We discuss more technical connections between these papers and ours later in the text.

Another related formalism is *perpetual voting* (Lackner 2020), which essentially corresponds to voting on multi-issue decisions in sequential order. In this setting, issues are chronologically ordered, i.e., decided one after the other. The work of Lackner (2020) and its follow-up by Lackner and Maly (Lackner and Maly 2021, 2023) do not consider strategic issues. Kozachinskiy et al. (2025) further study this setting, providing bounds on the dissatisfaction of voters. Moreover, Bulteau et al. (2021) move to a non-sequential (offline) model of perpetual voting and therein study proportional representation; we briefly consider this model in Section 6. In the same model, Elkind, Neoh, et al. (2024) study proportionality and strategyproofness for the utilitarian and egalitarian rules. Chandak et al. (2024) and Elkind, Obraztsova, J. Peters, et al. (2025) study proportionality in both the sequential and offline settings. Finally, note that perpetual voting and its offline variant have also been referred to as *temporal voting*; see the survey by Elkind, Obraztsova, and Teh (2024).

A third related formalism is that of *public decision making* (Conitzer, Freeman, et al. 2017; Fain et al. 2018; Freeman et al. 2017; Kahana and Hazon 2023). As in our model, public decision making considers k issues and for each one alternative has to be chosen. However, most public decision making models allow for more general utility functions than the simple approval preferences that we assume. One exception is the paper by Skowron and Górecki (2022), who introduce two proportional voting rules PAV and MES, in a setting that is essentially equivalent to ours. We study PAV, but leave a study of MES for future work as it is not defined when issues are decided sequentially, the scenario where free-riding might be the most appealing.

Our model is also related to *multi-winner voting* (Faliszewski et al. 2017; Lackner and Skowron 2023). The main difference is that instead of selecting k candidates from the same set of candidates, we have individual candidates for each of the k issues. In our paper, we adapt the class of Thiele rules from the multi-winner setting to ours. This class has been studied extensively, both axiomatically (Aziz, Brill, et al. 2017; Lackner and Skowron 2021; Sánchez-Fernández, Elkind, et al. 2017; Sánchez-Fernández and Fisteus 2019) and computationally (Aziz, Gaspers, et al. 2015; Brill, Gölz, et al. 2024; Godziszewski et al. 2021; Skowron, Faliszewski, et al. 2016). The concept of free-riding has also been considered for multi-winner elections (Hylland 1992; Schulze 2004). In particular, in the approval setting, authors have studied this by looking at *subset-manipulation*, i.e., manipulating by submitting only a subset of one's truly approved candidates (Botan 2021; D. Peters 2018). We note that, while this notion is related to ours in spirit, it is technically distinct: there is no requirement for the untruthfully-disapproved candidates to be in the truthful outcome. Subset-manipulation has also been studied in the setting of *fair mixing* (Brandl et al. 2021), a generalization of multi-winner voting where the output is a probability distribution over candidates.

In multi-winner voting, there is also substantial literature on the relationship between fairness (often proportionality) and strategyproofness (e.g., Kluiving et al. 2020; Lackner and Skowron 2018; Lee 2018; D. Peters 2018). Delemazure et al. (2022) study strategyproofness in the related model of *party-approval multi-winner voting*, which is a special case of our model (where alternatives and preferences are constant across issues).

To conclude, we note that free-riding is a very general phenomenon and has been widely studied in the economic literature on public goods (Groves and Ledyard 1977; Samuelson 1954) and philosophy (Tuck 2008). It has also been considered in more technical domains, such as free-riding in memory sharing (Friedman et al. 2019) and in peer-to-peer networks (Adar and Huberman 2000).

2 The Model

As is customary, we write $[k]$ to denote $\{1, \dots, k\}$.

We study a form of multi-issue decision making, where for each issue there are two or more possible options (candidates) available. We mostly focus on the sequential case where issues are decided one after the other. Furthermore, we assume that for each issue each voter submits an approval ballot, i.e., a subset of candidates that she likes. Formally, k denotes the number of issues and $C_1, \dots, C_k$ the respective sets of candidates. Let $N = [n]$ denote the set of voters. We write $A_i(v) \subseteq C_i$ for the approval ballot of voter v concerning issue i. In combination, we call such a triple $\mathcal{E} = \langle (C_i)_{i=1}^k,\ N,\ (A_i)_{i=1}^k \rangle$ an *election*. If k is clear from the context, we write $\bar{C}$ for $(C_i)_{i=1}^k$ and $\bar{A}$ for $(A_i)_{i=1}^k$.

A *partial outcome* of an election is an ℓ-tuple $\bar{w} = (w_1, \dots, w_\ell)$ with $w_i \in C_i$ and $\ell \in [k]$. If $\ell = k$, then we refer to $\bar{w}$ simply as an *outcome*. Given an election $\mathcal{E}$ and a (possibly partial) outcome $\bar{w}$ of length ℓ, the *satisfaction* of voter $v \in N$ with $\bar{w}$ is $sat_{\mathcal{E}}(v, \bar{w}) = |\{i \in [\ell] : w_i \in A_i(v)\}|$. In other words, the satisfaction of a voter is the number of issues that were decided in this voter's favour.[2] Furthermore, we write $s_{\mathcal{E}}(\bar{w}) = (s_1, \dots, s_n)$ to denote the n-tuple of satisfaction scores $(sat_{\mathcal{E}}(v, \bar{w}))_{v \in N}$ sorted in increasing order, i.e., $s_1 \leq s_2 \leq \cdots \leq s_n$. If the election $\mathcal{E}$ is clear from the context, we omit it in the notation.

Given an election $\mathcal{E}$ and a partial outcome for the first $\ell < k$ issues, a *sequential voting rule* returns the winner of the $(\ell + 1)$-th issue. A straightforward example is the *utilitarian rule* which returns, in each issue, the candidate with highest support (assuming some fixed tiebreaking among candidates). Note that this rule completely ignores the outcome of previous issues. We now introduce the two main classes of voting rules that we study, both of which include the utilitarian rule as a special case.

2.1 OWA Rules

OWA voting rules for multi-issue domains were proposed by Amanatidis et al. (2015) and are based on *ordered weighted averaging* operators (Yager 1988). We study them mostly in their sequential formulation. An OWA voting rule is defined by a family of vectors $\{\alpha^n\}_{n=1}^\infty$, where each $\alpha^n = (\alpha_1, \dots, \alpha_n)$ has length n and satisfies $\alpha_1 > 0$ and $\alpha_j \geq 0$ for $j \in [n]$. Given an election with n voters, the score of a (possibly partial) outcome $\bar{w}$ subject to α^n is

$$OWA_{\alpha^n}(\bar{w}) = \alpha^n \cdot s(\bar{w}),$$

[2]Note that different notions of satisfaction are possible; for instance, we could assume that voters have fine-grained preferences over issues and candidates. However, our simple model is a natural starting point, and we leave the investigation of different notions of satisfaction for future work.

where $\cdot$ is the scalar (dot) product. In issue i, given a partial outcome $\bar{w}$ for the first $i-1$ issues, the OWA rule returns the candidate $c \in C_i$ with maximum $OWA_{\alpha^n}((\bar{w}, c))$ (where $(\bar{w}, c)$ is the ℓ-tuple formed by appending c to $\bar{w}$). If more than one candidate achieves the maximum score, we use a fixed tiebreaking order among outcomes. We typically omit the superscript of α^n, as n is clear from the context. Note that the utilitarian rule corresponds to $\alpha^n = (1/n, \dots, 1/n)$. Another notable rule is the *egalitarian rule*, which corresponds to $\alpha^n = (1, 0, \dots, 0)$. This rule selects, at each step, the candidate that maximizes the satisfaction of the least-satisfied voter. Note that there are typically many such candidates, and thus this rule is rather indecisive. One can thus consider the *leximin* rule, which – among outcomes that are optimal for the least satisfied voter – maximizes the satisfaction of the second-least satisfied voter, then the third-least, etc.

Example 1. *Consider an election with 100 voters and 4 issues with the same three candidates, $\{a, b, c\}$. There are 66 voters that approve $\{a\}$ in all issues, 33 voters that approve $\{b\}$ in all issues, and one voter that approves always $\{c\}$. The utilitarian rule selects the outcome $\bar{w}_1 = (a, a, a, a)$ as a is the most approved candidate in each issue, achieving a total satisfaction of $\sum_{v \in N} sat(v, \bar{w}_1) = 4 \cdot 66$. The leximin rule selects $\bar{w}_2 = (a, b, c, a)$. Indeed, in the first round, all alternatives give a satisfaction of 0 to (at least) 34 voters; alternative a is the alternative that maximizes the satisfaction of the 35th least-satisfied voter. In the second round, all alternatives give satisfaction 0 to the least-satisfied voter; alternative b is the only alternative where the second least-satisfied voter has satisfaction of 1, and is hence selected. In the third round, c is the only alternative giving a satisfaction of at least 1 to everyone. The last round selects a, similarly as the first round. On the other hand, the egalitarian rule is highly indecisive in this election, as many different outcomes are optimal; depending on the tiebreaking rule, it can return, e.g., $\bar{w}_1 = (a, a, a, a)$, $\bar{w}_3 = (b, b, b, b)$ or even the questionable $\bar{w}_4 = (c, c, c, c)$.*

More precisely, the leximin rule is based on the leximin ordering $\succ$. Given two outcomes $\bar{w}$ and $\bar{w}'$ with $s(\bar{w}) = (s_1, \dots, s_n)$ and $s(\bar{w}') = (s'_1, \dots, s'_n)$, $\bar{w} \succ \bar{w}'$ if there exists an index $j \in [n]$ such that $s_1 = s'_1, \dots, s_{j-1} = s'_{j-1}$ and $s_j > s'_j$. At issue i, the *leximin rule* returns a candidate $c \in C_i$ such that $(\bar{w}, c)$ is maximal w.r.t. $\succ$. One can show that this rule corresponds to the OWA rule defined by the vector $\alpha^n = (1, 1/kn, 1/k^2n^2, \dots)$.

Proposition 1. *The OWA rule defined by $\alpha = (1, \frac{1}{kn}, \frac{1}{k^2n^2}, \dots)$ is equivalent to the leximin rule.*

Proof. Assume that $\bar{w} \succ \bar{w}'$, i.e., for $s(\bar{w}) = (s_1, \dots, s_n)$ and $s(\bar{w}') = (s'_1, \dots, s'_n)$, there exists an index $j \in [n]$ such that $s_1 = s'_1, \dots, s_{j-1} = s'_{j-1}$ and $s_j > s'_j$. Then

$$OWA_\alpha(\bar{w}) - OWA_\alpha(\bar{w}') = \alpha \cdot s(\bar{w}) - \alpha \cdot s(\bar{w}') =$$

$$\underbrace{(s_j - s'_j)}_{\geq 1} \cdot \frac{1}{(kn)^{j-1}} + \sum_{\ell=j+1}^{n} \underbrace{(s_\ell - s'_\ell)}_{\geq -k} \cdot \frac{1}{(kn)^{\ell-1}} \geq \frac{1}{(kn)^{j-1}} - k \sum_{\ell=j+1}^{n} \frac{1}{(kn)^{\ell-1}} \geq$$

$$\frac{1}{(kn)^{j-1}} - k(n-1)\frac{1}{(kn)^j} > 0.$$

This argument is symmetric in $\bar{w}$ and $\bar{w}'$, so we have shown that $\bar{w} \succ \bar{w}'$ iff $OWA_\alpha(\bar{w}) - OWA_\alpha(\bar{w}') > 0$. Thus, a maximal element with respect to $\succ$ achieves a maximum OWA_α-score and vice versa. ◻

2.2 Thiele Rules

The second class we study is based on Thiele methods (introduced by Thiele (1895), see the book by Lackner and Skowron (2023)). While Thiele methods are a class of multi-winner voting rules, they can be adapted to our setting straightforwardly. A voting rule in the Thiele class is defined by a function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ satisfying

$f(1) > 0$ and $f(\ell) \geq f(\ell + 1)$ for all $\ell \in \mathbb{N}$. The f-Thiele rule assigns a score of

$$\mathit{Thiele}_f(\bar{w}) = \sum_{v \in N} \sum_{\ell=1}^{sat(v,\bar{w})} f(\ell)$$

to a (partial) outcome $\bar{w}$. In each issue i, given a partial outcome $\bar{w}$, the rule returns the candidate $c \in C_i$ with maximum $\mathit{Thiele}_f((\bar{w}, c))$. Intuitively, these are weighted approval rules for which the weight assigned to each voter only depends on her satisfaction. Note that the utilitarian rule corresponds to $f_{util}(\ell) = 1$. The egalitarian rule does not appear in this class, whereas, for fixed n and k, leximin is equivalent to the Thiele method with $f_{lex}(\ell) = 1/(kn)^{\ell-1}$ (among other quickly decreasing functions). The most important Thiele rule is $f(\ell) = 1/\ell$, which is called *Proportional Approval Voting* in the multi-winner setting. We also refer to this Thiele rule as PAV.

Example 2. *Continuing Example 1, we see that for PAV in the first round a has maximum score of* 66. *In the second round, both partial outcomes* (a, a) *and* (a, b) *achieve the same score of* $66 + 33 = 99$. *The final outcome is either* $\bar{w}_5 = (a, b, a, a)$ *or* $\bar{w}_6 = (a, a, b, a)$*, depending on tiebreaking. Note that PAV is more majoritarian than leximin as it essentially ignores the single voter approving* $\{c\}$.

2.3 Global-Optimization-Based Rules

In Section 6, we will consider the setting where voters report their preferences over the issues at the same time, and all issues are decided simultaneously. In this setting, we look at voting rules based on a global optimization objective. That is, they select some outcome maximizing some score function. Given a sequential rule $\mathcal{R}$, we refer to its optimization-based counterpart (computed as explained below) as opt-$\mathcal{R}$.

The optimization-based rule corresponding to the OWA rule α^n simply selects the complete outcome maximizing $OWA_{\alpha^n}(\bar{w})$, whereas the rule corresponding to the Thiele rule f selects the outcome maximizing $\mathit{Thiele}_f(\bar{w})$. Again, we assume a fixed tiebreaking order over the outcomes. For example, opt-egalitarian selects an outcome $\bar{w}$ maximizing $\min_{v \in N} sat(v, \bar{w})$. This rule is NP-hard to compute (Amanatidis et al. 2015). In contrast, the opt-utilitarian rule is identical to its sequential formulation, and thus is poly-time computable. In this setting, we will often consider opt-leximin, which again is part of the opt-OWA family with $\alpha^n = (1, 1/kn, 1/k^2n^2, \ldots)$ (see Proposition 1). This rule selects any outcome $\bar{w}$ that is maximal w.r.t. the leximin ordering $\succ$, as previously defined.

We note that in this paper we focus on the sequential case, unless explicitly noted.

2.4 Free-Riding

In this paper, we study a specific form of strategic manipulation called *free-riding*. Intuitively, this means that a voter misrepresents her preferences on an issue where her favorite candidate wins also without her support. If the used voting rule takes the satisfaction of voters into account (as most OWA and Thiele methods do), such a manipulation can increase the voter's influence on other issues.

Example 3. *Consider an election with three voters and two issues. The first issue is uncontroversial: all voters approve candidate a. The second issue is highly controversial: all voters approve different candidates* ($A_2(1) = \{x\}$, $A_2(2) = \{y\}$, $A_2(3) = \{z\}$)*. If PAV (with alphanumeric tiebreaking) is used to determine the outcome, it could select, e.g., the outcome* (a, x)*. This leaves voters* 2 *and* 3 *less satisfied than voter* 1*. Both of them could free-ride to improve their satisfaction. Consider voter* 2*. If voter* 2 *changes her ballot on the first issue to another candidate, the outcome changes to* (a, y)*, as, according to the ballots, a yields a score of* 2 *in the first round, and* (a, y) *of* 3 *in the second round, which is optimal. As voter* 2*'s true preferences are positive towards a, this manipulation was successful.*

In the following, given an election $\mathcal{E}$ and a rule $\mathcal{R}$ such that $\mathcal{R}(\mathcal{E}) = (w_1, \ldots, w_k)$, we indicate w_i as $\mathcal{R}(\mathcal{E})_i$.

Definition 1. *Consider an election $\mathcal{E} = \langle (C_i)_{i=1}^k, N, (A_i)_{i=1}^k \rangle$, a voter $v \in N$ and a voting rule $\mathcal{R}$. Let $\mathcal{R}(\mathcal{E}) = (w_1, \ldots, w_k)$. We say that voter v can free-ride in election $\mathcal{E}$ on issues $I \subseteq [k]$ if there exists another election $\mathcal{E}^* = \langle (C_i)_{i=1}^k, N, (A_i^*)_{i=1}^k \rangle$ that only differs from $\mathcal{E}$ in the approvals of v for issues in I such that, for all $i \in I$, $w_i \in A_i(v)$, $w_i \notin A_i^*(v)$ and $\mathcal{R}(\mathcal{E}^*)_i = w_i$. In this case, we also say that v can free-ride in $\mathcal{E}$ via $\mathcal{E}^*$.*

Usually, we say a voter can manipulate if she can achieve a higher satisfaction by misrepresenting her preferences. In contrast, Definition 1 makes no assumptions about the satisfaction of the free-riding voter. Instead, we only require that the manipulator can misrepresent her preference in an issue without changing the outcome of the issue. This might lead to the same, a higher or lower satisfaction for the manipulator. This distinction will be crucial when talking about the risk of free-riding.

We say that *a voting rule $\mathcal{R}$ can be manipulated by free-riding* if there exists an election $\mathcal{E}$, a voter v and an election $\mathcal{E}^*$ such that v can perform free-riding in $\mathcal{E}$ via $\mathcal{E}^*$ and $sat_{\mathcal{E}}(v, \mathcal{R}(\mathcal{E})) < sat_{\mathcal{E}}(v, \mathcal{R}(\mathcal{E}^*))$.

Finally, note that sometimes we will consider a more general notion of manipulation, called *generalized* free-riding, where we lift the constraint that the outcome on the issues where free-riding occurs remains exactly the same. Most of our results hold for both models,[3] but in some cases we are able to get stronger results for the generalized case. This only plays a role in Section 4, where we study computational complexity. We defer its formal definition to that section.

3 Possibility and Risk of Free-Riding

In this section, we ask two main questions: The first question is for which voting rules free-riding is actually possible. We approach this question from two angles: First, we identify a small set of mild axioms and show that any rule satisfying these axioms is susceptible to free-riding. Secondly, we focus on the families of sequential rules we introduced in Section 2 and show that free-riding is possible for essentially all rules in these families. These two approaches complement each other, as they do not cover the same set of voting rules.

The second question is under which conditions free-riding may lead to a *decrease in satisfaction* of the free-riding voter. We analyze for which rules free-riding entails such a risk.

3.1 Possibility of Free-Riding

We first identify a multi-issue voting rule where free-riding is not possible: the utilitarian rule. Observe that with the utilitarian rule the outcomes for different issues do not influence each other. Thus, artificially decreasing one's satisfaction brings no advantage with this rule. Therefore, the utilitarian rule cannot be manipulated by free-riding.

Proposition 2. *The utilitarian rule cannot be manipulated by free-riding.*

Next, we show that for essentially all other reasonable rules, free-riding is possible. In order to formulate axioms that are as mild as possible, we will focus on a class of elections with a particularly simple structure. We claim that in such elections it is easy to argue about how a rule should behave.

Definition 2 (Simple elections)**.** *For every $k \geq 1$, $\ell \in [k]$ and $n \geq 2$, an* ℓ-simple (k, n)-election *satisfies the following:*

- *there are k issues and n voters;*
- *in each issue there are two candidates: $C_1 = \cdots = C_k = \{a, b\}$;*
- *every voter $v < n$ approves $\{a\}$ in every issue;*
- *voter n approves $\{b\}$ in ℓ issues, and $\{a\}$ in the remaining $k - \ell$.*

[3] The only exceptions are Propositions 14 and 15.

An example 2-simple $(6, 4)$-election is

$$\begin{aligned} A_1 &= (\{a\},\ \{a\},\ \{a\},\ \{a\},\ \{a\},\ \{a\}),\\ A_2 &= (\{a\},\ \{a\},\ \{a\},\ \{a\},\ \{a\},\ \{a\}),\\ A_3 &= (\{a\},\ \{a\},\ \{a\},\ \{a\},\ \{a\},\ \{a\}),\\ A_4 &= (\{b\},\ \{a\},\ \{a\},\ \{a\},\ \{b\},\ \{a\}). \end{aligned}$$

Note that for every number of issues $k \in \mathbb{N}$ and of voters $n \geq 3$, there is exactly one k-simple (k, n)-election (namely, the election where voter n always approves only b) and exactly k different 1-simple (k, n)-elections.

Given this definition, we introduce the following axioms for multi-issue voting rules.[4]

- **Near-unanimity**: For $n \geq 3$, the complete outcome of every 1-simple (k, n)-election must be $(a, \ldots, a)$.
- **Incentive for minorities**: For $n \geq 3$, there must be a number of issues k such that the complete outcome of the k-simple (k, n)-election is a permutation of $(a, \ldots, a, b)$.

The two axioms are logically independent; the utilitarian rule satisfies near-unanimity but not incentive for minorities, and vice versa for the constant rule always selecting $(b, a, a, \ldots)$. Both rules are also immune to free-riding.

Note that a weaker formulation of incentive for minorities could require for b to win *at least* once instead of *exactly* once. We have two arguments in favor of our stronger formulation. First, every sequential rule that satisfies the weaker variant of this axiom must also satisfy our stronger notion. Secondly, one can interpret the additional requirement of the stronger notion (where b must win exactly once, instead of at least once) as a continuity condition: if given $k - 1$ issues voter n does not deserve any representation (satisfaction 0), then they should not suddenly deserve to decide more than one issue when there is exactly one additional issue. Nevertheless, one could adopt this weaker notion instead and get results analogous to what we obtain in this section by slightly strengthening the other axioms. We do so in Appendix A.

First, we show that any rule that satisfies these two axioms is susceptible to free-riding.

Theorem 3. *Any rule that satisfies near-unanimity and incentive for minorities can be manipulated by free-riding.*

Proof. Consider a rule that satisfies the conditions of the theorem. Consider any $n \geq 3$ and let k be the number of issues whose existence is prescribed by incentive for minorities. Focus on the k-simple (k, n)-election $\mathcal{E}$ and let i be the unique issue where b is the outcome for $\mathcal{E}$. Consider now the 1-simple (k, n)-election $\mathcal{E}'$ where voter n approves of b in issue i. By near-unanimity, the outcome must be a in all issues. But then voter n can free-ride in $\mathcal{E}'$ via $\mathcal{E}$ in issues $[k] \setminus \{i\}$, completing the proof. □

Essentially, this result shows that as soon as we want to give *some* (but not all) power to minorities, free-riding becomes unavoidable.

Observe that the proof of Theorem 3 requires the manipulator to free-ride in multiple issues. If we additionally enforce the following axioms, then free-riding can occur even if we restrict the manipulator to only free-ride in one issue.

- **Issue-wise unanimity**: For every ℓ-simple (k, n)-election, if in some issue all voters approve of a, then a must be selected in that issue.
- **Monotonicity**: If the complete outcome of the k-simple (k, n)-election contains b exactly once, then for all $\ell \leq k$, the complete outcome of any ℓ-simple (k, n)-election contains b at most once.

[4]Observe that all the axioms we propose apply to both sequential and optimization-based rules. Moreover, notice that they only speak about binary elections (i.e., in every issue there are exactly two alternatives). Therefore, all our axiomatic results also hold for the special case of binary elections.

Note that adding issue-wise unanimity is only a mild requirement, here, as its core idea is already embodied by near-unanimity (although the two are logically independent).

Theorem 4. *Any rule that satisfies near-unanimity, incentive for minorities, issue-wise unanimity, and monotonicity can be manipulated by free-riding in one issue.*

Proof. Consider a rule that satisfies the conditions of the theorem. Consider any $n \geq 3$ and let k be the number of issues whose existence is prescribed by incentive for minorities. Let ℓ be the minimal ℓ for which there is an ℓ-simple (k, n)-election where b wins exactly once, and let $\mathcal{E}$ be such an election. Observe that such an ℓ must exist (by incentive for minorities), that we must have $\ell > 1$ (by near-unanimity), and that for all $(\ell - 1)$-simple (k, n)-elections b is not in the outcome (by monotonicity). Let i be the issue where b wins in $\mathcal{E}$; note that in this issue voter n must approve of b (by issue-wise unanimity). Next, let j be some issue distinct from i where voter n approves of b (such a j exists as $\ell > 1$). Let $\mathcal{E}'$ be the $(\ell - 1)$-simple (k, n)-election obtained by letting voter n approve of a instead of b in issue j. Observe that voter n can free-ride in $\mathcal{E}'$ via $\mathcal{E}$ in issue j, completing the proof. □

Observe that the previous results do not fully cover the families of rules we have introduced. For example, the egalitarian rule does not satisfy near-unanimity (the outcome might depend on the tiebreaking). Clearly, one might be concerned by what failing such mild axioms means for a rule; nonetheless, we now present a complementary result that shows that the whole set of families we have introduced can be manipulated via free-riding (excluding the utilitarian rule, as per Proposition 2).

Theorem 5. *Every sequential Thiele and sequential OWA rule except the utilitarian rule can be manipulated by free-riding.*

Proof. First, let $\mathcal{R}$ be a sequential f-Thiele Rule different from the utilitarian rule. Then, there exists a k such that $f(k-1) > f(k)$. Consider a $k+1$ issue election with four voters and two candidates a and b such that for the first k issues all voters only approve candidate a. Moreover, on issue $k + 1$ voters 1 and 2 approve b while voter 3 and 4 approve a. Assume further that a is preferred to b in the tiebreaking order. Clearly, a wins in the first k issues. Hence in issue $k + 1$ all voters have weight $f(k)$ which means both candidates have a score of $2f(k)$. By tiebreaking a wins. We claim that voter 1 can manipulate by changing her vote in one of the first k issues to $\{b\}$. Let i be the issue on which 1 manipulates. Then, in issue i, candidate a has a score of $3f(i-1)$ while b has a score of $f(i-1)$. Now, $f(i-1) > f(k)$ implies that $f(i-1) > 0$. Therefore $3f(i-1) > f(i-1)$, which means a still wins in issue i. It is clear that a also wins in the other issues until $k + 1$. In issue $k + 1$, a has a score of $2f(k)$ while b has a score of $f(k) + f(k-1)$. By assumption, this means that b wins on issue $k + 1$. Therefore, voter 1 did free-ride successfully.

Next, let $\mathcal{R}$ be a sequential OWA-Rule that is not the utilitarian rule. Consider an election with 2 issues and k voters. In each issue there are k candidates $a_1, \dots a_k$. In the first issue, voters 1 and 2 approve a_1. Every other voter $i \in \{3, \dots, k\}$ approves a_i. In the second issue voter 1 approves a_1, voter 2 approves a_2 and all other voters approve both a_1 and a_2. We assume that candidates with a lower index are preferred by the tiebreaking, which is applied lexicographically. Then there exists a k for which the vector α for k voters satisfies $\alpha_1 > \alpha_k$. In the first issue a_1 has to be selected, as no other candidate can have a higher OWA score. Then, as before (a_1, a_1) and (a_1, a_2) lead to the highest possible score on when looking at the second issue. By tiebreaking, (a_1, a_1) wins. Now, we claim that voter 2 can free-ride by approving a_2 instead of a_1 in the first issue. After the free-riding, all candidates are tied for the first issue, hence a_1 wins by tiebreaking. However, then following the discussion above, a_2 needs to be selected in the second issue. It follows that voter 2 did successfully free-ride. □

Hence, free-riding is essentially unavoidable if we want to guarantee fairer outcomes using Thiele or OWA rules.

3.2 The Risk of Free-Riding

Intuitively, free-riding seems to offer a simple and risk-free way to manipulate. However, we observe that for most sequential voting rules, free-riding may lead to a decrease in satisfaction. First, we can show that this holds for all sequential Thiele rules, except the utilitarian rule.

Proposition 6. *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function for which there is an $i \in \mathbb{N}$ such that $f(i) > f(i+1)$. Then, under the sequential f-Thiele rule, free-riding can reduce the satisfaction of the free-riding voter.*

Proof. Consider a sequential f-Thiele rule such that $f(i) > f(i+1)$ and consider the following election with nine voters, $i+4$ issues and candidates $a, \dots, g$ for all issues. We assume alphabetical tiebreaking. The approvals for each issue are given by these tuples:

$$\begin{aligned}
A_1 = \cdots = A_{i-1} &= (\{a\}, \{a\}, \{a\}, \{a\}, \{a\}, \{a\}, \{a\}, \{a\}, \{a\}) \\
A_i &= (\{a\}, \{a\}, \{a\}, \{b\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}) \\
A_{i+1} &= (\{b\}, \{a\}, \{c\}, \{b\}, \{b\}, \{a\}, \{a\}, \{a\}, \{b\}) \\
A_{i+2} &= (\{b\}, \{a\}, \{c\}, \{b\}, \{e\}, \{a\}, \{f\}, \{a, b\}, \{g\}) \\
A_{i+3} &= (\{b\}, \{c\}, \{d\}, \{e\}, \{b\}, \{f\}, \{a\}, \{a\}, \{g\})
\end{aligned}$$

Then, $\{a\}$ is clearly the winner for the first $i-1$ issues. Thus, all voters have a satisfaction of $i-1$ before issue i and a wins on issue i as it has the most supporters. In issue $i+1$, a and b increase the Thiele score by $f(i+1) + 3f(i)$, while c increases the score only by $f(i+1)$. By tiebreaking, a wins again. Then, in issue $i+2$, a increases the score by $f(i+2)+2f(i+1)$, b by $f(i)+2f(i+1)$, while all other candidates by at most $f(i)$. As $f(i) > f(i+1) \geq f(i+2)$ (together with the tiebreaking rule if $f(i+1) = 0$), it follows that b wins in issue $i+2$. Finally, in issue $i+3$, a increases the score by $f(i+1) + f(i+2)$, b by $f(i) + f(i+2)$ and all other candidates increase the score by at most $f(i)$. Hence b wins.

Now assume voter 1 changes her preferences and free-rides in issue i. It is straightforward to check that a remains the winner for issue i, but winner in issue $i+1$ changes to b while a now wins for issue $i+2$ and $i+3$. Therefore, 1 now additionally approves of the winner on issue $i+1$ but does not approve the winners of issues $i+2$ and $i+3$ any more. Hence, free-riding led to a lower satisfaction for the free-riding voter. □

The same holds for the following large class of sequential OWA rules.

Proposition 7. *Consider a sequential α-OWA rule such that there exists an $n \geq 8$ for which α^n is nonincreasing and satisfies $\alpha_3 > \alpha_{n-2}$. Then, free-riding can reduce the satisfaction of the free-riding voter.*

Proof. Consider an election with 4 issues and n voters. In each issue the candidate set is a subset of $\{a_1, \dots a_n\}$. The specific set of candidates is defined as all candidates that receive at least one approval according to the following description:

Issue 1: Voter $3, \dots n-3$ and n approve a_n. Every other voter i approves a_i.
Issue 2: Voter $1, 2, 3$ approve a_n, voters $n-2, n-1, n$ approve a_1. Every other voter i approves a_i.
Issue 3: Voter 1 and voter 4 approve a_4, voter $n-1$ and n approve a_n, Every other voter i approves a_i.
Issue 4: Voter 2 and 3 approve a_2, voter $n-2$ and n approve a_n and every other voter i approves a_i.

We assume that candidates with a higher index are preferred by the tiebreaking.

Let us determine the outcome of this election. In the first issue a_n has to be selected, as no other candidate can have a higher OWA score. This leads to satisfaction vector $(0, 0, 0, 0, 1, \dots, 1)$. In the second issue selecting a_1 or a_n both lead to a satisfaction vector of $(0, 0, 1, \dots, 1, 2)$. Selecting any other candidate leads to a satisfaction vector of $(0, 0, 0, 0, 1, \dots, 1, 2)$. Clearly, it is again the case that no candidate can have a higher OWA score than a_n. Hence a_n wins again.

In the third issue, selecting any candidate other than a_4, a_3, a_{n-2} or a_n leads to a satisfaction vector of $(0, 0, 1, \ldots, 1, 2, 2)$. Selecting a_3 leads to a satisfaction vector of $(0, 0, 1, \ldots, 1, 3)$, while selecting a_4 leads to a satisfaction vector of $(0, 0, 1, \ldots, 1, 2, 2, 2)$ and a_{n-2} to $(0, 1, \ldots, 1, 2)$. Finally, selecting a_n leads to a vector of $(0, 1, \ldots, 1, 2, 2)$. As $\alpha_2 \geq \alpha_3 > \alpha_{n-2} \geq \alpha_n$ selecting a_n leads to a higher OWA-score than selecting a candidate a_i with $i < n - 2$. Moreover, the OWA-score of a_{n-2} cannot be higher than the score of a_n. Hence, a_n wins in issue three.

In the fourth issue, selecting any candidate other than a_2 or a_n leads to a satisfaction vector of $(0, 1, \ldots, 1, 2, 2, 2)$. Selecting a_2 leads to a satisfaction vector of $(0, 1, \ldots, 1, 2, 2, 3)$, while selecting a_n leads to a satisfaction vector of $(1, \ldots, 1, 2, 3)$. As $\alpha_1 \geq \alpha_3 > \alpha_{n-2}$ selecting a_n leads to the highest OWA score.

Now, we claim that voter n can free-ride by approving any other candidate in the first issue. Indeed, if voter n approves any other candidate it is still the case that no candidate can have a higher OWA score than a_n. Let us consider how the other issues change:

In the second issue selecting a_1 leads to a satisfaction vector of $(0, 0, 1, \ldots, 1)$. Selecting a_n leads to a satisfaction vector of $(0, 0, 0, 1, \ldots, 1, 2)$, while selecting any other candidate leads to a satisfaction vector of $(0, 0, 0, 0, 0, 1, \ldots, 1, 2)$. As $\alpha_3 > \alpha_n$ we know that the OWA score of a_1 is higher than that of a_n which is at least as high as the OWA score of every other candidate.

In the third issue, selecting any candidate other than a_4, a_2 or a_n leads to a satisfaction vector of $(0, 0, 1, \ldots, 1, 2)$. Selecting a_2 leads to a satisfaction vector of $(0, 1, \ldots, 1)$, while selecting a_4 leads to a satisfaction vector of $(0, 1, \ldots, 1, 2)$. Finally, selecting a_n leads to a vector of $(0, 0, 1, \ldots, 1, 2, 2)$. As $\alpha_2 \geq \alpha_3 > \alpha_{n-1}$ selecting a_4 leads to a higher OWA-score than selecting a candidate a_i with $i \neq 2, 4$. Moreover, the OWA-score of a_2 cannot be higher than the score of a_4. Hence, a_4 wins in issue three.

In the fourth issue, selecting any candidate other than a_2, a_4, or a_n leads to a satisfaction vector of $(0, 1, \ldots, 1, 2, 2)$. Selecting a_2 leads to a satisfaction vector of $(1, \ldots, 1, 2, 2)$, while selecting a_4 leads to a satisfaction vector of $(0, 1, \ldots, 1, 3)$ and a_n to a satisfaction vector of $(0, 1, \ldots, 1, 2, 2, 2)$. As $\alpha_1 > \alpha_{n-2}$ selecting a_2 leads to the highest OWA score. However, this decreases the satisfaction of voter n with respect to the honest ballots to 2. □

Note that the conditions of Proposition 7 do not include the sequential egalitarian rule, but an equivalent statement holds nonetheless:

Proposition 8. *Free-riding can decrease the satisfaction of the free-riding voter under the sequential egalitarian rule.*

Proof. Consider an election with 5 issues and 5 voters. In each issue there are 3 candidates a, b and c. We assume that tiebreaking always prefers a. The approval sets are given as follows:

	Issue 1	Issue 2	Issue 3	Issue 4	Issue 5
Voter 1	$\{b\}$	$\{a\}$	$\{b\}$	$\{b\}$	$\{b\}$
Voter 2	$\{b\}$	$\{a\}$	$\{a\}$	$\{b\}$	$\{a\}$
Voter 3	$\{b\}$	$\{a\}$	$\{a\}$	$\{c\}$	$\{b\}$
Voter 4	$\{a\}$	$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$
Voter 5	$\{a\}$	$\{a\}$	$\{b\}$	$\{a\}$	$\{b\}$

Let us determine the winners under the sequential egalitarian rule: In the first issue, every option leads to a minimal satisfaction of 0. Therefore, a is winning by tiebreaking. Then, in the second issue, a must be winning as it raises the minimal satisfaction to 1. In the third issue, again, no alternative can increase the minimal satisfaction and therefore a wins by tiebreaking. This leads to a situation where every voter except 1 has a satisfaction of 2 while 1 has a satisfaction of 1. Hence, in issue four, b must be the winner, as it increases the minimal satisfaction to 2. Finally, in the fifth issue, electing b leads to a minimal satisfaction of 3, which is better than electing a. We observe that voter 1 has a satisfaction of 3 in the end.

Now, we claim that voter 1 can free-ride on issue two. If voter 1 approves b instead, then all candidates lead to the same minimal satisfaction of 0. Hence, a wins by tiebreaking. If voter 1 decides to free-ride on issue two, this changes the winners of the following issues as follows: In issue three, b must now be the winner, as it increases the minimal satisfaction to 1. Then, in issue four and five, no candidate increases the minimal satisfaction and hence a wins both issues by tiebreaking. However, this decreases the satisfaction of voter 1 with respect to the honest ballots, to 2. □

4 Computational Complexity

In this section, we will study the computational complexity of successful free-riding: that is, raising one's total satisfaction via free-riding. Overall, we will show that it is generally hard to do so, even with full information. Observe that, due to the performance of, e.g., modern SAT- or ILP-solvers, computational hardness (in particular NP-completeness) cannot be seen as an unbreakable defense against manipulation. However, the main appeal of free-riding is its simplicity. A manipulator that is able to solve computationally hard problems (and has full information about other's preferences) has no benefit from restricting the potential manipulation to free-riding – such a voter could optimize their satisfaction via arbitrary manipulation.

Observe that the outcome of a sequential rule is always polynomial-time computable: for every round, we can just iterate over all the candidates involved in that issue and pick the one maximizing the score.

Therefore, it is computationally easy to decide if (some way of) free-riding is possible: for each issue where a voter approves of the winner c, one can just iterate over all singleton ballots $\{c'\}$ with $c' \neq c$ and check if under any such ballot c still wins.[5]

However, although voters can easily verify if free-riding is possible, it might be hard to judge its long-term consequences. If this is unfeasible, voters might be discouraged from free-riding (because, as we have shown, it can have negative consequences). Hence, we study the following problem:

Successful-$\mathcal{R}$-Free-Riding
Input: An election $\mathcal{E} = \langle N, \bar{A}, \bar{C} \rangle$ and a voter $v \in N$.
Question: Is there an election $\mathcal{E}^*$ such that v can free-ride in $\mathcal{E}$ via $\mathcal{E}^*$ and $sat_{\mathcal{E}}(v, \mathcal{R}(\mathcal{E})) < sat_{\mathcal{E}}(v, \mathcal{R}(\mathcal{E}^*))$?

First, we show that free-riding is NP-complete for a large class of sequential f-Thiele rules and the sequential egalitarian rule.

Theorem 9. *Successful-$\mathcal{R}$-Free-Riding is NP-complete for every sequential f-Thiele rule for which there exists an $\ell \in \mathbb{N}$ such that (i) for all $j, j' \in [\ell]$ it holds $f(j) = f(j')$ and (ii) f is strictly decreasing on $\mathbb{N} \setminus [\ell - 1]$.*

The conditions of this theorem apply to all functions that are constant up to a certain number ℓ, and from ℓ on become strictly decreasing. This is the case, e.g., for the sequential PAV rule.

Proof of Theorem 9. Fix an f-Thiele rule $\mathcal{R}$ satisfying the conditions of the theorem. First, notice that Successful-$\mathcal{R}$-Free-Riding is in NP, as, given an insincere approval ballot for the manipulator, we can check whether it improves her satisfaction in polynomial time and whether it is a case of free-riding.

Now, we show hardness by a reduction from 3-SAT (Garey and Johnson 1979). Let ϕ be a 3-CNF with n variables and m clauses. We refer to the j-th clause as C_j. We assume w.l.o.g. that ϕ is not satisfied by setting all variables to false and that each clause contains exactly three literals. Furthermore, let $k \in \mathbb{N}$ be the smallest k such that

[5]Note that restricting oneself to singleton ballots is safe under OWA and Thiele rules: it is easy to see that, if there exists a ballot $A_i^*(v)$ via which voter v can free-ride in issue i, then she can free-ride in the same issue also via any ballot in the set $\{\{c'\} : c' \in A_i^*(v)\}$.

$f(k+1) < f(k)$. As $\mathcal{R}$ is not the utilitarian rule, such a k surely exists, and is constant in the size of ϕ (as f does not depend on the number of either the voters or the issues). Moreover, let $\ell \in \mathbb{N}$ be the smallest ℓ such that $f(k+1)(\ell+1) < f(k)\ell$. Again, note that ℓ can be large, but does not depend on the instance. We will construct an instance of SUCCESSFUL-$\mathcal{R}$-FREE-RIDING with $3n(\ell+1)+5$ voters and $k+3n$ rounds.

For each variable x_i, we have three voters s_i, v_i and $\bar{v}_i$, 3ℓ voters $r_1^i, t_1^i, w_1^i, \dots, r_\ell^i, t_\ell^i, w_\ell^i$. Furthermore, we have four additional voters a, u_1, u_2, u_3. Finally, we have a distinct voter v, who will try to free-ride.

In the first $k-1$ rounds, there are two candidates, c and $\bar{c}$, and every voter approves of both candidates. Here, v cannot increase her satisfaction by manipulating, and all voters win in each round. Thus, the satisfaction of every voter after the first $k-1$ rounds will be exactly $k-1$.

Now, focus on rounds from k to $k+3n-1$. We subdivide this set of rounds into triples; that is, the first triple is $k, k+1, k+2$, the second $k+3, k+4, k+5$, and so on. We refer to the j-th round of triple i as round (i,j); for example, round $(2,3)$ corresponds to round $k+5$. In the first round of every triple, there is one candidate c, plus one candidate c_{v^*} for every voter $v^* \in N$. In the second and third rounds, we additionally have a voter $\bar{c}$. We assume that, if there is a tie, c wins against every voter-candidate, but loses against $\bar{c}$.

For every triple i, in round $(i,1)$, voters a, v, s_i approve of candidate c. Every other voter approves only of her voter-candidate. In round $(i,2)$, voters $v, v_i, r_1^i, \dots, r_\ell^i$ and $a, \bar{v}_i, t_1^i, \dots, t_\ell^i$ vote for c and $\bar{c}$, respectively. The rest of the voters vote for their voter-candidate. In round $(i,3)$, voters $v, t_1^i, \dots, t_\ell^i$ and $w_1^i, \dots, w_\ell^i$ vote for c and $\bar{c}$, respectively. Again, the rest of the voters vote for their voter-candidate.

First, note that whoever votes for a voter-candidate never wins. We show this by induction. In round k, every voter has satisfaction $k-1$, and hence the candidates approved most often will win; this cannot be any voter-candidate, as c is the most approved. Now, suppose this holds up to a round i. In round $i+1$, we know that there is at least one voter voting for c that has never won since round $k-1$, as she always voted for her voter-candidate up to round i; for example, voters s_i, v_i or w_1^i in the cases where round $i+1$ is the first, second or third round of the triple, respectively. Such voters contribute to the score of c with a value of $f(k-1)$. Since no voter-candidate can have a score greater than $f(k-1)$, our claim follows.

Consequently, in every round $(i,1)$, voter v can free-ride by voting for her voter-candidate. We claim that if v free-rides in $(i,1)$ then she wins in round $(i,2)$ and loses in round $(i,3)$; if she does not, the opposite happens. Furthermore, we claim that at the beginning of every triple, v and a have the same satisfaction. We show both claims by induction.

Consider round $(1,1)$. If v does not free-ride, c wins, and the satisfaction of a and v will be k. Now consider round $(1,2)$. The approval score of both c and $\bar{c}$ is $f(k+1)+f(k)+f(k)\ell$, and hence $\bar{c}$ wins by tiebreaking. Thus, in the next round, the approval score of c will be $f(k)\ell + f(k+1)$, and the score of $\bar{c}$ will be $f(k)\ell$; hence, c wins. Now, suppose that v free-rides. Then, her satisfaction after round $(1,1)$ will be $k-1$. Hence, the approval score of c would increase to $2f(k)+f(k)\ell$, making it the winner. Hence, in round $(1,3)$ the scores of c and $\bar{c}$ would be $f(k+1)(\ell+1)$ and $f(k)\ell$, respectively. As we assume $f(k+1)(\ell+1) < f(k)\ell$, here we have that $\bar{c}$ wins, as desired. Observe that in both cases v and a won the same number of rounds.

Now suppose the claim holds up to triple i. Then, let s be the satisfaction of v and a at the beginning of triple $i+1$. Observe that if v does not free-ride in $(i+1,1)$, then in round $(i+1,2)$ the approval score of c and $\bar{c}$ is $f(s+2)+f(k)+f(k)\ell$, and hence $\bar{c}$ wins by tiebreaking. Thus, in round $(i+1,3)$, the score of c will be $f(k)\ell + f(s+2)$, and of the score of $\bar{c}$ will be $f(k)\ell$; hence, c wins. If v does free-ride, then in round $(i+1,2)$ the approval score of c raises to $f(s+1)+f(k)+f(k)\ell$, making it the winner. Furthermore, in round $(i+1,3)$, the scores of c and $\bar{c}$ would be $f(k+1)\ell + f(s+2)$ and $f(k)\ell$, respectively. Clearly, $\bar{c}$ wins here. Observe again that v and a won the same number of rounds.

Let us move to the final round. Here, there are $m+1$ candidates, namely $c, c_1, \dots, c_m$. Here, each voter v_i votes for c_j if $x_i \in C_j$ (and similarly for $\bar{v}_i$ and $\bar{x}_i$). Furthermore, v, u_1, u_2, u_3 vote for c, voter a votes for all candidates except for c, and everyone else votes for all candidates. If there is a tie, c loses against every other candidate. We can interpret c (resp. $\bar{c}$) winning in round $(i, 2)$ as setting x_i to true (resp. false). We claim that this assignment satisfies ϕ if and only if c wins in the final round.

Indeed, let α be the score contributed by the voters who vote for all candidates. Furthermore, let β be the score contributed by v or by a, which are the same (as shown before). Observe that u_1, u_2, u_3 won exactly $k-1$ rounds. Furthermore, v_i won exactly k rounds if c won in round $(i, 2)$, and $k-1$ otherwise (and conversely for $\bar{v}_i$ and $\bar{c}$).

Thus, the score of c in the final round will be $\alpha + \beta + 3f(k)$. Furthermore, given a clause C_j, if all of its literals are unsatisfied, the score of c_j will also be $\alpha + \beta + 3f(k)$. By our rule on tiebreaking, here c_j wins. If, on the other hand, some literals in C_j are satisfied, the score of c_j will be at most $\alpha + \beta + 2f(k) + f(k+1)$. Hence, if all clauses are satisfied, c wins, as desired.

Now, observe that v can free-ride only in every first round of every triple, but not elsewhere. Indeed, in the first $k-1$ rounds, whatever v votes for will be the winner. Furthermore, in the second and third rounds of every triple, v is either losing (and hence cannot free-ride) or her weight is breaking a tie between c and some other candidate (which means that, if she were to vote for some other candidate instead, c would no longer win). Observe also that, as shown earlier, v will win all the $k-1$ first rounds, plus two rounds per triple (irrespective of whether she free-rides or not). Therefore, the only way that v can raise her satisfaction is by making c win in the last round by forcing a satisfying assignment for ϕ by free-riding. It follows that v can free-ride if and only if ϕ is satisfiable, and we are done. ◻

We further consider the egalitarian rule.

Theorem 10. *Successful-$\mathcal{R}$-Free-Riding is* NP*-complete for the sequential egalitarian rule.*

The proof of Theorem 10 can be found in Appendix B.

Now, we will consider a weaker notion of manipulation, called *generalized free-riding*. This allows us to get results for a broader class of rules, in particular for OWA rules. Here, we assume that a voter can free-ride if she can change her ballot so that she still approves of the winning alternative (though this alternative might differ from the truthful winner). That is, we just require that the new winning candidate is still (truthfully) approved by the manipulator. At its core, generalized free-riding is based on the assumption that voters are indifferent between approved candidates. To formally define generalized free-riding, we replace $\mathcal{R}(\mathcal{E}^*)_i = w_i$ in Definition 1 with $\mathcal{R}(\mathcal{E}^*)_i \in A_i(v)$.

Remark. *Note that the definition of generalized free-riding is considerably more permissive than the main definition (Definition 1), as we allow $\mathcal{R}(\mathcal{E}^*)_i$ to belong to $A^*_i(v)$. We nevertheless consider this notion because it is technically related to our main concept and enables us to derive results that hold in a broader setting.*

The problem of Generalized-Successful-$\mathcal{R}$-Free-Riding is then defined analogously to Successful-$\mathcal{R}$-Free-Riding.

Theorem 11. *Generalized-Successful-$\mathcal{R}$-Free-Riding is* NP*-complete for every sequential f-Thiele rule distinct from the utilitarian rule such that $f(i) > 0$ holds for every $i \in \mathbb{N}$.*

Proof. Fix an f-Thiele rule $\mathcal{R}$ satisfying the conditions of the theorem. First, notice that Generalized-Successful-$\mathcal{R}$-Free-Riding is in NP, as we can guess an insincere approval ballot for the manipulator and check whether it improves her satisfaction (and is an instance of generalized free-riding) in polynomial time.

Now we show hardness by a reduction from 3-SAT. In the following, recall that, in every round i, a voter v gives each of her approved candidates an extra score of $f(sat(v, \bar{w}_1^{i-1}) + 1)$, where $\bar{w}_1^{i-1} = w_1, \dots w_{i-1}$, and the candidate with the highest score wins.

Then, let ϕ be a 3-CNF with n variables and m clauses. We assume w.l.o.g. that ϕ is not satisfied by setting all variables to true and that each clause contains exactly three literals. Furthermore, let $k \in \mathbb{N}$ be the smallest k

such that $f(k+1) < f(k)$. As $\mathcal{R}$ is not the utilitarian rule, such a k indeed exists and is constant in the size of ϕ. We construct a GENERALIZED-SUCCESSFUL-$\mathcal{R}$-FREE-RIDING instance with $k+n$ rounds as follows: For every variable x_i there are 4 voters $v_i^1, v_i^2, \bar{v}_i^1$ and $\bar{v}_i^2$. Furthermore, we add nine voters $v_0^1, v_0^2, w, u_1, \dots, u_6$. Finally, we add another voter v, who will be the distinguished voter that tries to manipulate. In the first $k-1$ rounds, there are two candidates c and $\bar{c}$. In the following n rounds, there are three candidates c_0, c and $\bar{c}$ plus one candidate c_{v^*} for every $v^* \in N$. In round $k+n$, we have $m+1$ candidates $c, c_1, c_2 \dots, c_m$ plus one candidate c_{v^*} for every $v^* \in N$. We assume that if ties need to be broken between c_0 and another candidate, then c_0 is selected and if a tie between c and a candidate other than c_0 needs to be broken, then c wins.

In the first $k-1$ rounds, all voters approve both candidates. Hence, v cannot increase her satisfaction by manipulating, and all voters win in each round. Thus, the satisfaction of every voter after the first $k-1$ rounds will be exactly $k-1$.

We continue with n rounds such that, in round i, v_0^1 and v_0^2 approve c_0, v_i^1 and v_i^2 approve c, $\bar{v}_i^1$ and $\bar{v}_i^2$ approve $\bar{c}$, v approves both c and $\bar{c}$ and w approves $c, \bar{c}$ and c_0. Finally, all other voters $v^* \in N$ only approve their candidate c_{v^*} except in round k, where u_1 and u_2 additionally approve c_0, c and $\bar{c}$.

Then, in round $k+n$, v and $u_1, \dots, u_6$ approve c. Furthermore, for every candidate c_i with $1 \le i \le m$, $\bar{v}_j^1$ and $\bar{v}_j^2$ approve c_i if and only if variable x_j appears positively in C_i and v_j^1 and v_j^2 approve c_i if and only if variable x_j appears negatively in C_i. Additionally, w approves $c_1, \dots, c_m$. All other voters approve only their candidate.

We claim that in rounds k to $k+n-1$ *(i)* c wins if v approves c and $\bar{c}$ (i.e., v does not misrepresent her preferences), *(ii)* either c or $\bar{c}$ becomes the winner if v approves only one of these candidates, and *(iii)* c_0 wins if v approves neither c nor $\bar{c}$. We show the claim by induction. Up until round $k-1$, all voters have gathered the same satisfaction $k-1$, and hence in round k each voter contributes to the score of their approved candidates with the same value of $f(k)$. The only candidates that are approved by more than one voter are c_0, c and $\bar{c}$, where c_0 is approved by five voters, while c and $\bar{c}$ are both approved by six voters. Therefore, by our assumption about tiebreaking, c is the winner in round k. Furthermore, if v misrepresents her preferences and votes only for either c (resp. $\bar{c}$), then c (resp. $\bar{c}$) is the unique winner in round k. Finally, if v approves neither c nor $\bar{c}$, then c_0 wins by our assumption on tiebreaking. Observe that v is not allowed to make c_0 win, by the definition of generalized free-riding.

Now assume the claim holds for rounds $k, \dots i-1$. Then, in round i the only candidates that are approved by more than one voter are again c_0, c and $\bar{c}$. To be more precise, c_0 is approved by v_0^1, v_0^2 and w, c is approved by v_i^1, v_i^2, w and v and $\bar{c}$ by $\bar{v}_i^1$, $\bar{v}_i^2$, w and v. Crucially, v_0^1, v_0^2, v_i^1, v_i^2, $\bar{v}_i^1$ and $\bar{v}_i^2$ have not won in any of rounds $k, \dots i-1$, as they never approved of c or $\bar{c}$ in these rounds; hence, their satisfaction at this point is still $k-1$. Now, let s_v^{i-1} and s_w^{i-1} be the satisfaction of v and w up to round round $i-1$, respectively. Then, both c and $\bar{c}$ have a score of $2f(k) + f(s_v^{i-1}+1) + f(s_w^{i-1}+1)$, c_0 has score $2f(k) + f(s_w^{i-1}+1)$ whereas all other alternatives can have a maximal score of $f(k)$. As we know that $f(s_v^{i-1}+1) > 0$, this implies, by our assumption about tiebreaking, that c is the winner in round i. Furthermore, as before, if v misrepresents her preferences, she can make c resp. $\bar{c}$ the unique winner and if she approves neither c nor $\bar{c}$, then c_0 wins by our assumption on tiebreaking. Observe again that c_0 cannot win here, by definition of generalized free-riding.

We can interpret the winners in the rounds $k, \dots k+n-1$ as a truth assignment T by setting x_i to true if c wins in round i and to false if $\bar{c}$ wins in round i (observe that c_0 can never win by the previous arguments). Then, we claim that c wins in round $k+n$ if and only if C_j is satisfied by this truth assignment: All voter-candidates c_{v^*} are approved by at most one voter with satisfaction $k-1$, and hence have a score of at most $f(k)$. The satisfaction of v_j^1 and v_j^2 is $k-1$ if x_j is set to false in T and k if x_j is set to true in T. Similarly, the satisfaction of $\bar{v}_j^1$ and $\bar{v}_j^2$ is $k-1$ if x_j is set to true in T and k if x_j is set to false in T. Finally, the satisfaction of w is $k+n-1$. Hence, the approval score of c_i is $6f(k) + f(k+n)$ if all literals in C_j are set to false and at most $4f(k) + 2f(k+1) + f(k+n)$ if at least one literal is set to true. On the other hand, the satisfaction of u_1 and u_2 is k, the satisfaction of $u_3, \dots, u_6$ is

$k-1$ and the satisfaction of v is $k+n-1$. Hence, the approval score of c is $4f(k)+2f(k+1)+f(k+n)$. As we assumed $f(k+1) < f(k)$, we get that $4f(k)+2f(k+1)+f(k+n) < 6f(k)+f(k+n)$. Therefore, if there is a clause C_i for which no literal is set to true, then c_i has a higher approval score than c and hence, c is not a winner in round $k+n$. On the other hand, if for every clause at least one literal is set to true, then $c_1,\dots,c_m$ have at most the same score as c and c wins by tiebreaking.

Now, the honest ballot of v leads to the truth assignment in which every variable is set to true by tiebreaking. By assumption, this assignment does not satisfy ϕ and hence c does not win in round $k+n$. By construction, the satisfaction of v equals $k+n-1$ in this case. Moreover, as v cannot manipulate in the first $k-1$ rounds, the only way that v can gain more satisfaction is by forcing the winners in rounds $k,\dots k+n-1$ to form a satisfying truth assignment without allowing c_0 to win any round. Hence, v can manipulate via generalized free-riding if and only if ϕ is satisfiable. □

Theorem 12. *Generalized-Successful-$\mathcal{R}$-Free-Riding is* NP*-complete for every sequential α-OWA rule such that, for all n, $\alpha=(\alpha_1,\dots,\alpha_n)$ is nonincreasing and $\alpha_1 > \alpha_n$.*

Observe that we give three different reductions, depending on the form of the OWA vector. We present here one case; the others can be found in Appendix B.

Proof of Theorem 12. Fix an α-OWA rule $\mathcal{R}$ satisfying the conditions of the theorem. First, notice that Generalized-Successful-$\mathcal{R}$-Free-Riding is in NP, as we can guess an insincere approval ballot for the manipulator and check whether it improves her satisfaction (and is an instance of generalized free-riding) in polynomial time.

Next, we show hardness by a reduction from 3-SAT. Let ϕ be a 3-CNF with n variables and m clauses. We refer to the j-th clause as C_j. We assume w.l.o.g. that ϕ is not satisfied by setting all variables to false and that each clause contains exactly three literals. We will construct an instance of Generalized-Successful-$\mathcal{R}$-Free-Riding with $2n+5$ voters. More specifically, there are two voters v_i and $\bar{v}_i$ for each variable x_i, four voters $u_1,\dots,u_4$, plus one distinguished voter v who will try to manipulate.

Given the weight vector $\alpha=(\alpha_1,\dots,\alpha_{2n+5})$, we distinguish three (not necessarily exclusive) cases: (1) $\alpha_2 > \alpha_{n+5}$, (2) $\alpha_{n+1} > \alpha_{2n+5}$, and (3) $\alpha_2 = \alpha_{2n+5}$. Since $\alpha_1 > \alpha_{2n+5}$, at least one case must be true. In the following, we will give a reduction for the first case ($\alpha_2 > \alpha_{n+5}$). The other cases are similar; we discuss them in Appendix B.

We construct an instance with $n+2$ rounds as follows. In the first $n+1$ rounds, there are two candidates: c and $\bar{c}$. In the last round, there are $m+1$ candidates, namely $c, c_1,\dots,c_m$. We assume that, in the case of ties, c always loses.

In the first round, everybody votes for $\bar{c}$ except for v and u_1, who vote for c. Here, $\bar{c}$ wins by tiebreaking, and v cannot manipulate.

In each round i with $i\in\{2,\dots,n+1\}$, voter v_i votes for candidate $\bar{c}$, voter $\bar{v}_i$ for candidate c; everyone else votes for both candidates. We claim that v can manipulate in every such round i and force the win of either c or $\bar{c}$. We show so by induction. In round 2, suppose that v votes only for c (the case where she votes for $\bar{c}$ is analogous). Then, if c were to win, the satisfaction vector would be of form $(1,1,1,2,\dots,2)$ (everyone but v_1 wins). On the other hand, if $\bar{c}$ wins, then it would be of form $(0,1,1,2,\dots,2)$ (everyone but $\bar{v}_1$ and v win). Hence, c wins. Observe that if v votes truthfully, $\bar{c}$ wins by tiebreaking. Now, suppose this holds up to a round i. Before round $i+1$, v has won $i-1$ rounds (all but the first one), whereas voters $u_1,\dots,u_4$, as well as any pair of voters $v_j,\bar{v}_j$ (with $j\geq i$) have won i rounds. Furthermore, for every pair $v_j,\bar{v}_j$ (with $j<i$), exactly one voter won $i-1$ rounds while the other i rounds. Suppose again that v votes for c (the case where she votes for $\bar{c}$ is similar). Then, observe that, if c or $\bar{c}$ win, the satisfaction vectors (excluding v) would be completely symmetric (and every voter would have at least a score of i); however, if $\bar{c}$ wins, v would have a satisfaction of $i-1$, whereas if c wins, she would get a satisfaction of i. Hence, since $\alpha_1 > 0$, here c wins. Observe again that if v votes truthfully, then $\bar{c}$ wins.

Consider the final round. Up to here, v and u_1 have won n rounds (they lost the first round), while u_2, u_3, u_4 have won $n+1$ rounds. Furthermore, every voter v_i has won n rounds if c won in round $i+1$ and $n+1$ times otherwise (and conversely for $\bar{v}_i$ and $\bar{c}$). In this round, voters $u_1, \ldots, u_4$ approve of all candidates but c, voter v_i (resp. $\bar{v}_i$) approves of c and every candidate c_j such that $x_i \notin C_j$ (resp. $\bar{x}_i \notin C_j$). Finally, voter v approves of c. Observe that we can interpret c winning in round $i+1$ as setting x_i to true, and $\bar{c}$ winning as setting x_i to false. We claim that c wins in the last round if and only if this assignment satisfies ϕ. To see this, consider that, if c were to win, the satisfaction vector would be:

$$(n, \underbrace{n+1, \ldots, n+1}_{n+4 \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n \text{ times}}).$$

Let us call this vector s. Consider now a candidate c_j and its corresponding clause C_j. If all three of its literals are unsatisfied, then the corresponding voters all have satisfaction $n+1$. Hence, if c_j were to win in this case, the satisfaction vector would again be exactly s. By our assumptions on tiebreaking, here c_j would win against c. Furthermore, suppose that either one, two, or three of the literals have been satisfied. Then, the vectors are, respectively:

$$(n, n, \underbrace{n+1, \ldots, n+1}_{n+2 \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n+1 \text{ times}})$$
$$(n, n, n, \underbrace{n+1, \ldots, n+1}_{n \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n+2 \text{ times}})$$
$$(n, n, n, n, \underbrace{n+1, \ldots, n+1}_{n-2 \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n+3 \text{ times}})$$

Let these vectors be s_1, s_2 and s_3, respectively. One can show that if $\alpha_2 > \alpha_{n+5}$ the dot product between α and each of these three vectors would be strictly lower than the dot product between α and s. For example:

$$\begin{aligned} s \cdot \alpha > s_1 \cdot \alpha \iff \alpha_1 n + \left(\sum_{i=2}^{n+5} \alpha_i (n+1)\right) + \left(\sum_{i=n+6}^{2n+5} \alpha_i (n+2)\right) > \\ (\alpha_1 + \alpha_2) n + \left(\sum_{i=3}^{n+4} \alpha_i (n+1)\right) + \left(\sum_{i=n+5}^{2n+5} \alpha_i (n+2)\right) \iff \\ (\alpha_2 + \alpha_{n+5})(n+1) > \alpha_2 n + \alpha_{n+5}(n+2) \iff \alpha_2 > \alpha_{n+5}. \end{aligned}$$

The other two cases are similar. Hence, if C_j is satisfied, candidate c_j cannot win against c. Consequently, if all clauses are satisfied, candidate c wins.

Now, if c wins in the last round, then the satisfaction of v would be $n+1$; if c loses, it would be n. Notice also that v cannot raise her satisfaction by manipulating in the final round. Furthermore, if v always submits her true preferences, then by tiebreaking $\bar{c}$ would win in every round i with $i \in \{2, \ldots, n+1\}$. By assumption, this would not satisfy ϕ, and hence c would not win in the last round. Therefore, v has an incentive to manipulate in these rounds to try and choose a satisfying assignment for ϕ. It follows that v can manipulate via generalized free-riding if and only if ϕ is satisfiable, so we are done. □

5 Numerical Simulations

So far, we have seen that sequential Thiele and OWA rules are generally susceptible to free-riding. However, we have also seen that free-riding can be detrimental to free-riders, i.e., their satisfaction can decrease. In this section, we use numerical simulations[6] to shed more light on the *possibility, effect, and risk* of free-riding with sequential rules. We answer three main questions:

(1) How often do voters have the possibility to increase their (final) satisfaction by free-riding? How many instances contain issues in which free-riding leads to a lower satisfaction for the free-rider?
(2) What is the average risk of free-riding? That is, what is the likelihood of free-riding resulting in a negative outcome?
(3) Is there a difference in the effect of free-riding if it is done by a voter belonging to a majority or minority?

To be able to give robust answers, we study these questions in a range of different models.

5.1 Model Setup and Parameter Values

For generating ballots, we use the popular 2d-Euclidean model (Bredereck et al. 2019; Elkind, Faliszewski, et al. 2017; Enelow and Hinich 1990; Godziszewski et al. 2021; Szufa et al. 2025). In this model, both voters and candidates are points in a 2-dimensional space. For each issue, we sample candidate points from a uniform distribution on the $[-1, 1] \times [-1, 1]$ square. That is, candidates are different in each issue, independent from each other and across issues. In contrast, voters' points are the same for all issues. We consider three distributions for voters:[7]

- *square*: A uniform distribution on the $[-1, 1] \times [-1, 1]$ square, i.e., voters and candidates are independently drawn from the same distribution.
- *many groups*: Voters are split into four groups, the first three comprising each of 20%, the last of 40% of voters. Each group is centered around a different point, namely one of $(\pm 0.5, \pm 0.5)$. Voters' x and y coordinates are drawn independently from a normal distribution with standard deviation 0.2.
- *unbalanced*: Here there are two disjoint groups, one comprising of 20% of voters, the other of 80%. As before, the voters' locations are sampled from a normal distribution, with centers $(0.5, 0.5)$ and $(-0.5, -0.5)$, respectively, and standard deviation 0.1.

We transform these coordinates into approval ballots as follows: A voter approves the closest candidate as well as any candidate that is similarly close (within 1.5 times the distance).

Our default choice are multi-issue elections with $n = 20$ voters, $k = 20$ issues, and 5 candidates per issue. Later on we investigate the effect of this parameter choices. Finally, we sample 2000 elections for each voter distribution and choice of parameters.

5.2 Considered Voting Rules and Forms of Free-Riding

In our experiments, we consider a subclass of Thiele methods and a subclass of OWA rules. For better comparison, we parameterize both classes with a parameter x (albeit this parameter has a different interpretation in both classes). We consider α-OWA rules with

$$\alpha_x = (\underbrace{1, \dots, 1}_{n-x \text{ many}}, \frac{1}{kn}, \frac{1}{k^2 n^2}, \dots) \qquad \text{for } x \in \{0, 1, \dots, n-1\}.$$

Note that for decreasing x, more and more voters receive full consideration (weight 1). For $x = 0$, all voters receive full consideration and we obtain the utilitarian rule. In contrast, for $x = n - 1$, only the least satisfied voter receives full consideration; this is the sequential leximin rule (cf. Proposition 1).

[6] The source code as well as additional plots are available at https://github.com/martinlackner/free-riding.

[7] The unbalanced and many groups scenarios have been proposed by Chandak et al. (2024). We use the same parameter values to increase comparability, even though their election model is different from ours.

Further, we consider f-Thiele rules with

$$f_x(i) = \frac{1}{i^x} \qquad \text{for } x \in \{0, 0.25, 0.5, 0.75, \dots\}.$$

Note that for $x = 0$ this is the utilitarian rule, for $x = 1$ it is sequential PAV, and for increasing x it approaches the sequential leximin rule, since f_x diminishes quickly.

We distinguish two ways in which free-riding can occur. We speak of *single-issue free-riding* if a voter may free-ride once in a given election. In contrast, with *repeated free-riding* a voter free-rides whenever it is possible in a given election. Single-issue free-riding models situations where free-riding occurs rarely, e.g., if there is limited information available about expected outcomes, or if there is a high social cost for free-riding. In contrast, repeated free-riding models situations where free-riding is a viable strategy and and a free-riding voter can always exploit free-riding opportunities.

Note that we always assume that there is only one free-rider among all voters. We leave the study of free-riding group effects as a topic for future work.

5.3 Results

5.3.1 Possibility and risk of free-riding. In the single-issue free-riding model, we speak of *successful free-riding* for a given election, voter, and issue, if the voter can free-ride in the given issue and this increases her overall satisfaction (i.e., at the end of the election); we speak of *harmful free-riding* if the voter can free-ride but this decreases her satisfaction. Note that free-riding can also be neutral (with no change in satisfaction). In the repeated setting, we speak of *successful free-riding* for a given election and voter if the voter free-rides whenever possible and this increases her overall satisfaction; we speak of *harmful free-riding* if this decreases her satisfaction.

Let us first consider the single-issue free-riding model. That is, for each multi-issue election, we iterate over all voters and all issues and check whether free-riding is possible (Definition 1). Figure 1 shows the proportion of instances where in at least one issue successful/harmful free-riding is possible (averaged over all voters). (As successful and harmful free-riding may be possible in the same election − albeit in different issues − the proportion of successful and harmful free-riding may be larger than 1.) In addition, it shows the risk of free-riding: we define the risk of a voter in an election as the number of issues where harmful free-riding occurs divided by the number of issues where either successful or harmful free-riding occurs.

Let us discuss the results from Figure 1. First, we see that the results for the *square* and *many groups* distributions are very similar. In contrast, the *unbalanced* distribution yields different results. For all distributions, we clearly see that rules closer to the utilitarian rule ($x = 0$) are less susceptible to free-riding than those closer to leximin (larger values of x). We also see that − as expected − the utilitarian rule is the only rule where free-riding is not possible (cf. Proposition 2). We note that this increase in susceptibility (with distance to the utilitarian rule) has also been observed by Barrot et al. (2017) for arbitrary manipulations. Both the proportion of voters that can successfully free-ride and those with the possibility of harmful free-riding grow with parameter x.

The most important conclusion from this experiment is that the risk of single-issue free-riding is non-negligible: for the square/many groups/unbalanced distributions, the risks are 2.7%/4.0%/0.7% for PAV (Thiele, $x = 1$), and 16.5%/16.2%/8.4% for leximin (OWA, $x = 19$). This shows that harmful free-riding is not merely a theoretical possibility. In particular for more egalitarian voting rules (larger x-values in Figure 1), this risk could indeed decrease the temptation of free-riding.

If we move to repeated free-riding, the definition of risk changes: here, the risk of a voter in a given election is either 0 (the outcome of repeated free-riding is positive), or 1 (the outcome is negative). Risk is averaged over all voters (for whom successful or harmful free-riding is possible) and over all elections. We see that the risk is small compared to the single-issue model. This is because the probability of successful free-riding in a single issue is generally larger than harmful free-riding (i.e., risk in the single-issue model is ≤ 0.5), the risk almost

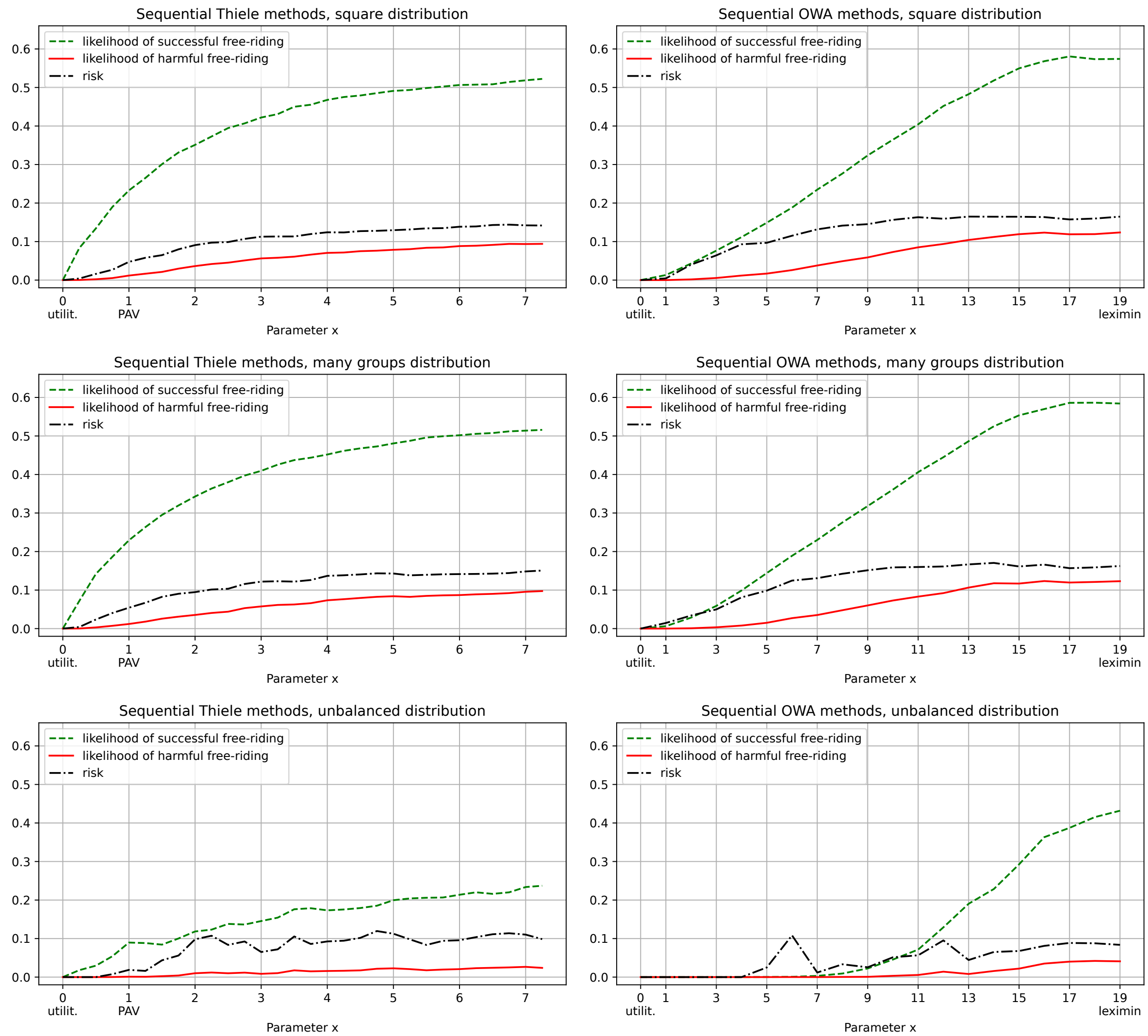

Fig. 1. Possibility and risk of single-issue free-riding. The left column of diagrams show sequential Thiele methods, the right sequential OWA methods. The three rows correspond to the *square*, *many groups*, and *unbalanced* distributions.

vanishes with sufficient repetitions. Figure 2 shows exemplarily the results for the *square* distribution. We see much higher likelihood of successful free-riding with almost no risk involved. Recall, however, that this form of free-riding requires the capability to reliably identify free-riding possibilities and to repeatedly exploit them.

So far, we have not taken into account how much a voter can benefit from free-riding. To investigate this, we show in Figure 3 the expected change in satisfaction for a free-riding voter. As expected, repeated free-riding leads to a more pronounced change in satisfaction. Furthermore, the change in satisfaction grows with parameter x, i.e., with increasing distance to the utilitarian rule. This aligns with the increased possibility to free-ride (as we have seen in Figures 1 and 2). Finally, note that in all cases, the change in satisfaction is not particularly large

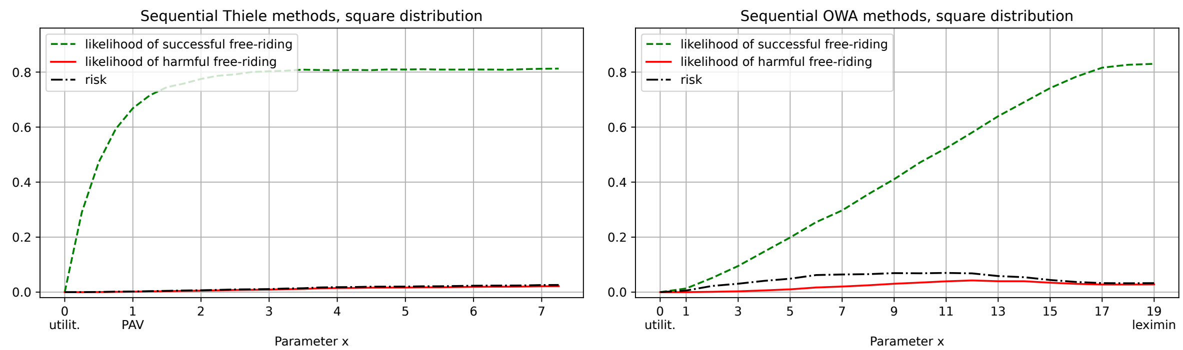

Fig. 2. Results for repeated free-riding with the *square* voter distribution. In contrast to Figure 1, we see a much higher likelihood of successful free-riding with very little risk.

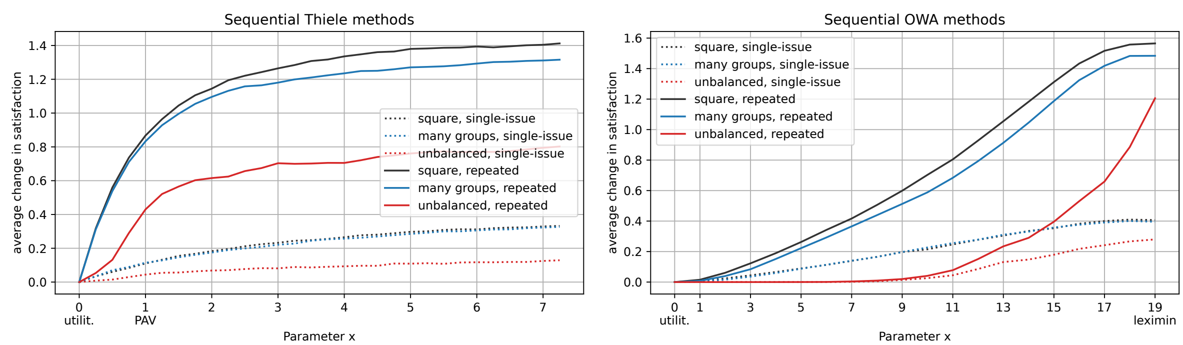

Fig. 3. Average change in satisfaction from free-riding, comparing different voter distributions as well as single-issue vs. repeated free-riding.

in comparison to the number of issues (20). Even with repeated free-riding when using the leximin rule, the expected number issues with an improvement is less than 1.6.

5.3.2 Impact of model parameters. Let us now briefly describe the impact of our chosen model parameters. In the following, we use our default setting ($n = 20$ voters, $k = 20$ issues, 5 candidates per issue, *square* distribution, single-issue free-riding) and vary a single parameter to observe its impact.

Increasing the number of voters (Figure 4) decreases the chance of voters being pivotal. In line with this observation, we see an overall decrease in both successful and harmful free-riding with an increase in voters. For a larger number of voters, it would make sense to move to a model where groups of voters free-ride. This requires additional assumptions about voter coordination (cf. the framework of iterative voting (Meir 2017)).

In Figure 5, we see that varying the number of candidates (per issue) has moderate impact. The increase in the likelihood of both successful and harmful free-riding can be explained by the increase in heterogeneity: as voters' preferences become more diverse and like-minded groups become smaller, a single free-riding action is more likely to change the outcome.

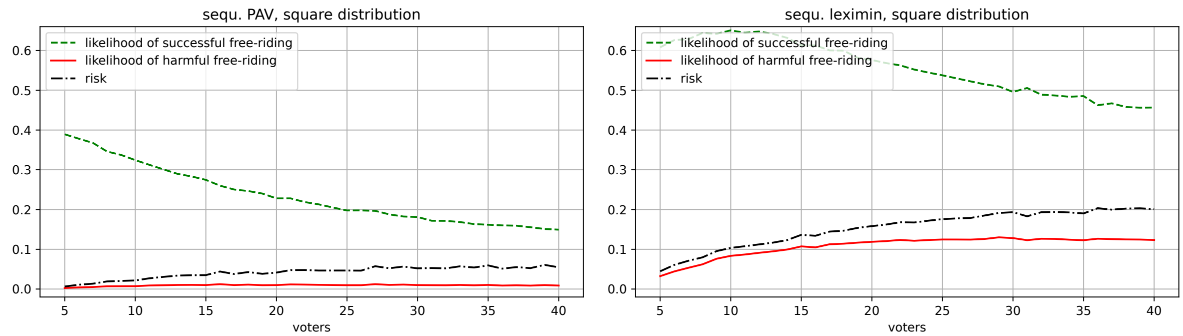

Fig. 4. Varying the number of voters (our default choice is $n = 20$): an increase in the number of voters decreases the effectiveness of free-riding by a single voter.

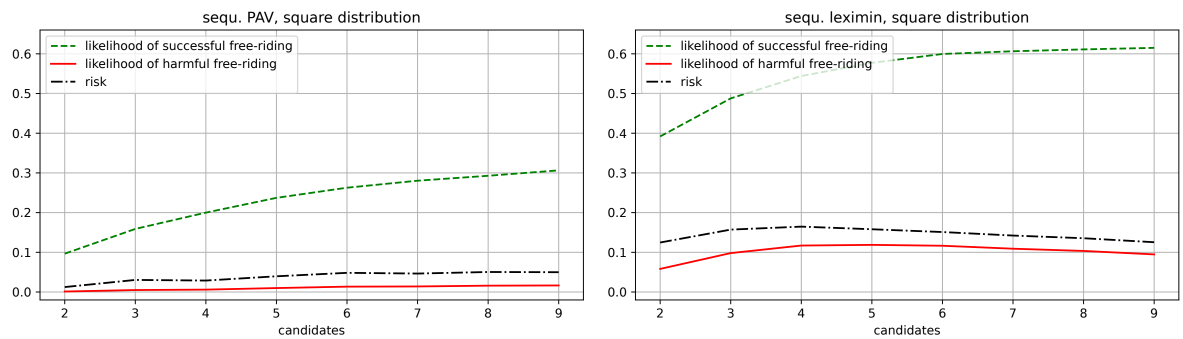

Fig. 5. Varying the number of candidates (our default choice is $m = 5$): more candidates per issue increase the likelihood of free-riding.

Finally, increasing the number of issues also increases the possibility of both successful and harmful free-riding, as some effects may only materialize in the long run. Note that also the risk of free-riding increases in a longer sequence of issues, in particular for sequential OWA methods.

5.3.3 Minority vs. majority free-riders. As a final experiment, we are studying the differences between free-riding voters belonging a minority or majority. Here, we use the *repeated free-riding* model, as it is generally more beneficial for the free-rider (as we have discussed in Section 5.3.1) and thus magnifies differences between free-riding and truthful voters. Let us first consider the many groups distribution. We randomly select one voter from the 40% group as "majority" voter, and one from a 20% group as "minority" voter. Figure 7 (upper part for *many groups*) shows that a majority voter is slightly more likely to successfully free-ride. This can be explained intuitively by the interpretation that free-riding majority voters form their own (one-voter) minority and thus are able to benefit from more egalitarian rules. However, when considering the average change in satisfaction for free-riding voters (Figure 8, upper part for *many groups*), we see that the impact of free-riding is small: the change in satisfaction for a free-riding voter (both minority and minority) is small in comparison to their total satisfaction.

The results for the unbalanced distribution are very similar in terms of satisfaction (Figure 8, lower part). A noteworthy difference is that – for Thiele rules – minority voters have a slightly larger change of successful

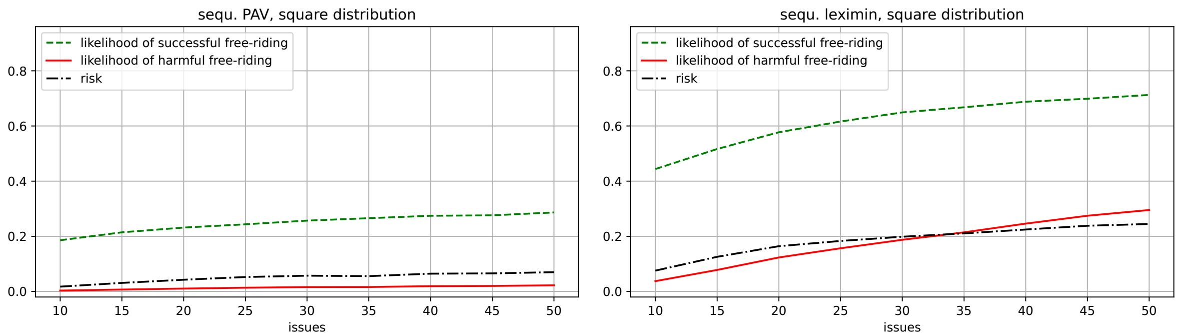

Fig. 6. Varying the number of issues (our default choice is $k = 20$): a longer sequence of decisions increases the potential impact of free-riding.

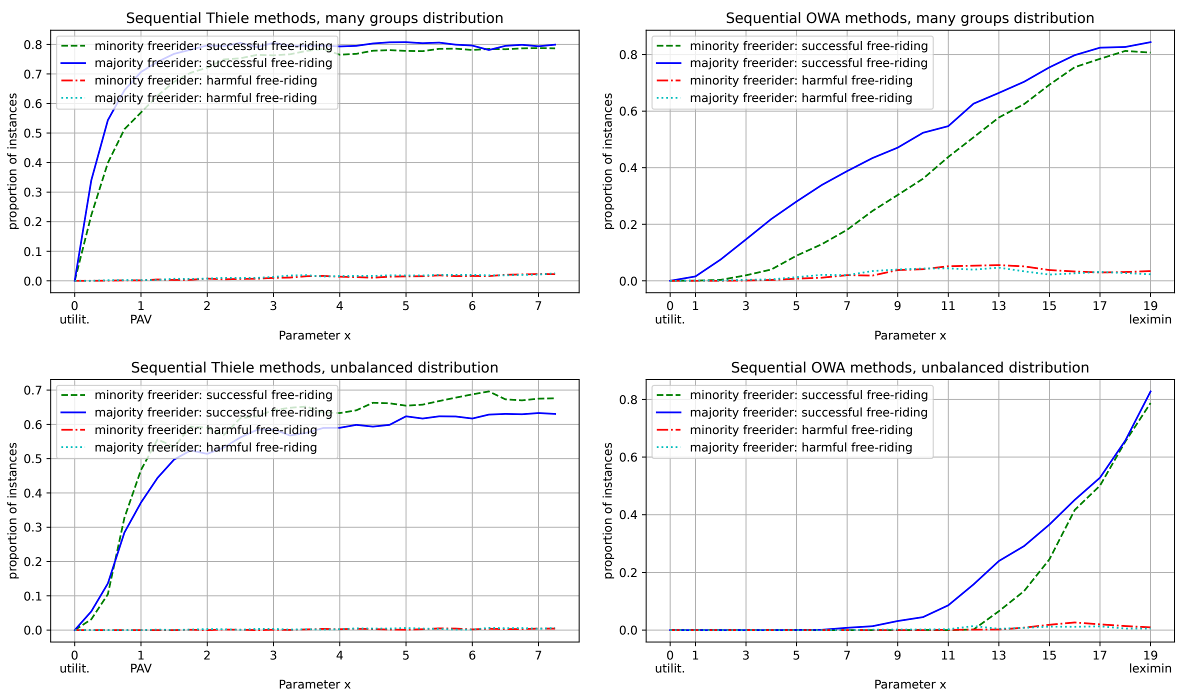

Fig. 7. The possibility of successful and harmful free-riding for minority and majority voters.

free-riding. This is probably due to the small weight that a single majority voter has within the majority block (16 voters in total). A second interesting observation is that free-riding is here almost non-existent for close-to-utilitarian OWA rules (Figure 7, lower part). This is due to the fact that for these rules the majority almost always dictates the decision and consequently free-riding is without effect.

Overall, we can conclude that for highly structured instances (such as the many groups and unbalanced distributions) free-riding is not an overly important phenomenon, due to its very limited impact on satisfaction.

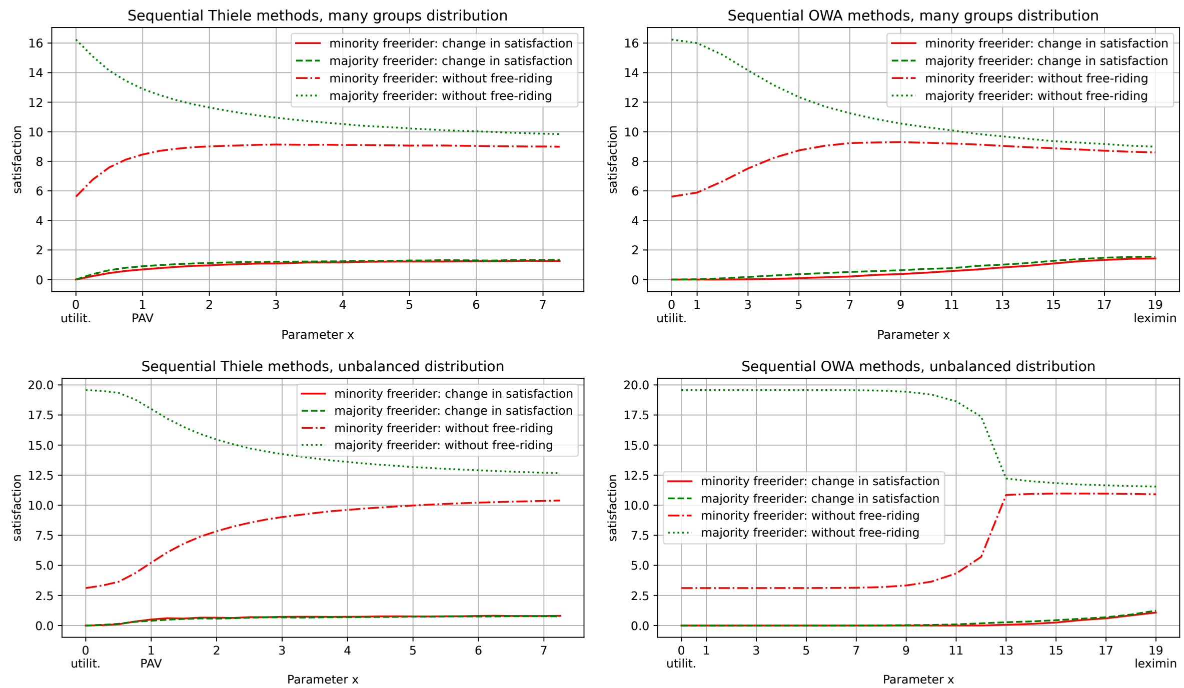

Fig. 8. The change in satisfaction for minority and minority voters in relation to the average satisfaction of voters in either groups *without* free-riding. We see that the change in satisfaction is similar for both groups (and relatively small).

This statement, however, only holds for single-voter free-riding. In this paper, we are not studying models for groups of free-riders. In contrast, we expect that group free-riding has significant impact in these scenarios (depending on the chosen model), in particular if a majority subgroup forms a “fake minority”. Note that a group free-riding model will require additional assumptions on information exchange between free-riders (is group free-riding even possible?) and group cohesiveness (are group preferences identical or merely similar?).

6 Free-Riding with Optimization-Based Rules

We conclude this paper by briefly looking at the case of optimization-based rules. Recall that, in this setting, voters submit their preferences for all issues at the same time, and the result is computed simultaneously to maximize some global objective. We defer the proofs of this section to Appendix C.

We believe that free-riding in this setting, as a phenomenon, is quite distinct from the sequential case. In the sequential case, issues are presented one by one, and at each step, a voter may choose to free-ride *in the current issue*. If a voter believes that a candidate she approves has a strong chance of victory (i.e., it has a large margin of support), she might be tempted to free-ride to immediately lower her (perceived) satisfaction. Our results show that it is computationally easy to free-ride but hard to determine whether it is possible to improve one’s satisfaction – even with full information. Moreover, in many settings, it is risky to do so with no information regarding future issues.

In contrast, free-riding in the optimization-based setting is more akin to classical, unrestricted manipulation. Given the interconnections among issues, each voter here must, assuming full information, carefully consider how to change her ballots to decide whether free-riding is possible, let alone beneficial. Indeed, as we will see, the main barrier against free-riding in the optimization-based case is the hardness of determining the winner (or the consequences of changing one's ballot). Thus, it seems free-riding here loses its simple, "greedy" appeal. Restricting oneself to free-riding in this setting is not advantageous to a manipulator. While it reduces the strategy space, it is not significantly easier than arbitrary manipulation (i.e., to change the full ballot with the goal to increase satisfaction).

We start by studying the possibility and risk of free-riding. Given that our axioms from Section 3 also apply to this setting, our results already indicate that, for essentially all rules that include some form of fairness towards minorities, free-riding is possible. However, not all the rules we study satisfy those axioms (again, this is the case of opt-egalitarian). Thus, similarly to Theorem 5, we integrate those results with the following.

Theorem 13. *Every opt-Thiele and opt-OWA rule except the utilitarian rule can be manipulated by free-riding.*

Interestingly, in contrast to the sequential case, for all opt-Thiele rules and for at least opt-leximin we find that the satisfaction of the free-riding voter can never decrease.[8]

Proposition 14. *Free-riding cannot reduce the satisfaction of the free-riding voter when an opt-Thiele rule is used, but it can increase the satisfaction of the free-riding voter.*

Proposition 15. *Free-riding cannot reduce the satisfaction of the free-riding voter when opt-leximin is used, but it can increase the satisfaction of the free-riding voter.*

(It remains an open problem to generalize the latter result to other opt-OWA rules.) The previous two results seem to indicate that free-riding is safe in this setting: however, we will now show that free-riding is hard to perform *at all*, i.e., it is hard to decide whether free-riding is possible in the first place. Hence, a manipulator has no reason to restrict herself to free-riding.

Towards our goal, we start from a more fundamental problem: outcome determination. Indeed, any hypothetical free-rider needs to decide if, by voting dishonestly, the outcome would be better than the "truthful" outcome. To do so, she must be able to determine the outcome of an election. If this step turns out to be intractable, then we already have a first computational barrier against free-riding. Hence, we study the following problem:

$\mathcal{R}$-Outcome-Determination
Input: An election $\mathcal{E} = \langle N, \bar{A}, \bar{C} \rangle$, an issue i and a candidate $c \in C_i$.
Question: Does c win in issue i under $\mathcal{R}$?

In the following, we assume that for all opt-f-Thiele rules, f is poly-time computable.[9] Similarly, we assume that, for a given opt-α-OWA rule and n voters, we can retrieve α^n in polynomial time. Now, we show that outcome determination is hard for both families of rules.

Theorem 16. *$\mathcal{R}$-Outcome-Determination is* NP*-hard for every opt-f-Thiele rule distinct from the utilitarian rule.*

To prove the above we reduce from CubicVertexCover (Alimonti and Kann 2000), similarly as in the proof of Theorem 3 by Skowron, Faliszewski, et al. (2016). Their result is similar to ours: they show NP-hardness for a large class of "OWA rules", which however correspond to what is commonly called "Thiele rules" in multi-winner voting (Lackner and Skowron 2023). We note that our result is not a consequence of theirs, since multi-winner voting is not a special case of our model.

[8]Propositions 14 and 15 do not hold for generalized free-riding. Indeed, consider a situation with two voters and two issues, where $A_1(v_1) = \{a, b\}$, $A_2(v_1) = \{a\}$, and $A_1(v_2) = A_2(v_2) = \{b\}$. Here, most reasonable rules (with alphabetical tiebreaking) pick (b, a). If v_1 changes her ballot in issue 1 to $A_1^*(v_1) = \{a\}$, the outcome (again, for most reasonable rules) becomes (a, b). This is an instance of generalized free-riding, but the final outcome is worse for v_1.
[9]This is justified by the fact that all relevant values of f can be computed ahead of time and stored in a look-up table.

Moreover, we note that the hardness of opt-PAV was already shown by Brill, Gölz, et al. (2024, Theorem 5.1), as the party-approval model used in this paper is a special case of ours. This implies that the hardness of opt-PAV holds even in the special case where the alternatives and preferences of the voters are constant across issues. Our result, in contrast, requires different preferences and candidates for each issue, but holds for all opt-f-Thiele rules except the utilitarian rule.

Theorem 17. *$\mathcal{R}$-Outcome-Determination is* NP*-hard for every opt-α-OWA rule such that, for all n, α^n is nonincreasing and $\alpha_1 > \alpha_n$.*

Note that our result is related to Theorems 2 and 3 by Amanatidis et al. (2015). While their results hold even for the special case of binary elections, they only focus on a special subclass of OWA rules, i.e., given by vectors of the form $(1, \ldots, 1, 0, \ldots, 0)$.

In light of the above, one could conclude that free-riding is unfeasible for optimization-based rules. Still, one could argue – especially since we use worst-case complexity analysis – that sometimes the fact that a certain candidate wins can still be known (or guessed). For example, when a candidate receives an extremely disproportionate support, or when some external source (i.e., a polling agency having the computational power to solve $\mathcal{R}$-Outcome-Determination) communicates the projected winners. In this case, the manipulator would need to solve a slightly different, potentially easier, problem: *Given that some candidate that I approve of wins in this specific issue, can I deviate from my honest approval ballot, without making this candidate lose*? Motivated by this, we study the following problem:

$\mathcal{R}$-Free-Riding-Recognition
Input: An election $\mathcal{E} = \langle N, \bar{A}, \bar{C}\rangle$, an issue i, a candidate $c \in C_i$ such that $c \in \mathcal{R}(\mathcal{E})_i$, and a voter v such that $c \in A_i(v)$.
Question: Can v free-ride in $\mathcal{E}$ on issue i?

We define Generalized-$\mathcal{R}$-Free-Riding-Recognition analogously. Luckily, the picture does not change: this problem is still computationally hard for essentially the same families of rules.

Theorem 18. *(Generalized-)$\mathcal{R}$-Free-Riding Recognition is* NP*-hard for every f-Thiele rule distinct from the utilitarian rule.*

Hardness for OWA rules is split in two theorems: one showing NP-hardness, the other coNP-hardness.

Theorem 19. *(Generalized-)$\mathcal{R}$-Free-Riding Recognition is* NP*-hard for every α-OWA rule for which there is a $c \geq 3$ such that, for every $n \in \mathbb{N}$, there is a nonincreasing vector α of size ℓ (with $3n \leq \ell \leq cn$) such that $\alpha_1 > \alpha_\ell$ and $\alpha_{3n} > 0$.*

Theorem 20. *(Generalized-)$\mathcal{R}$-Free-Riding Recognition is* coNP*-hard for every α-OWA rule for which there is a $c \geq 2$ such that, for every $n \in \mathbb{N}$, there is a nonincreasing vector α of size ℓ (with $n < \ell \leq cn$) such that $\alpha_1 > \alpha_\ell$ and $\alpha_{\ell-n+1} = 0$.*

Theorems 18, 19 and 20 strengthen our previous observations. We conclude that free-riding is generally unattractive for optimization-based rules, since the manipulator cannot even decide efficiently whether free-riding is possible.

7 Discussion and Research Directions

In this paper, we have demonstrated that free-riding is an essentially unavoidable phenomenon in sequential multi-issue voting (cf. Theorem 3 and other results in Section 3.1). However, we have also identified computational challenges for voters attempting to assess the actual consequences of free-riding. In addition, our numerical simulations show that the possibility of harmful free-riding is non-negligible, that is, a free-riding voter may end up with fewer satisfactory issues than without free-riding. This holds especially for voters who will not or cannot free-ride whenever they have the chance to – but only occasionally. Such reluctance may be caused by the social context in which free-riding occurs: in small groups, it may be obvious to other group members that free-riding takes place and thus can entail negative social consequences. Finally, our results show that the expected change in satisfaction of a free-riding voter is rather limited (albeit always positive in expectation). All in all, we conclude that free-riding in real-world applications of multi-issue decision making may be less attractive to voters than the theoretical possibility would initially suggest.

We conclude this paper with specific open problems. First, we would like to point out that many of our hardness proofs use several candidates per issue. Do all of these results still hold for binary elections? Second, our classification of sequential OWA rules with potentially harmful free-riding is not complete. Are there sequential OWA rules where free-riding is never harmful except for the utilitarian rule? Third, Theorem 10 shows hardness of Successful-$\mathcal{R}$-Free-Riding for the sequential egalitarian rule. Can this result be extended to a larger class of OWA rules? Fourth, there are further voting rules to be considered, such as rules based on Phragmén's ideas (Brill, Freeman, et al. 2017; Phragmén 1895). Finally, coordinated free-riding of groups of voters may be much more impactful than single-voter free-riding. Studying this phenomenon provides an attractive opportunity to study group dynamics in voting related to coordination and strategy selection.

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A Additional Results: Possibility of Free-Riding

In this section, we provide additional results complementing those in Section 3.1. As mentioned in that section, we consider the following weaker variant of incentive for minorities:

- **Weak incentive for minorities**: For $n \geq 3$, there must be a number of issues k such that the complete outcome of the k-simple (k, n)-election contains both a and b at least once.

We get results analogous to Theorems 3 and 4 via this weaker variant by strengthening the other axioms:

- **Strong near-unanimity**: For $n \geq 3$, the complete outcome of any ℓ-simple (k, n)-election can contain b at most $\ell - 1$ times.
- **Strong monotonicity**: For any $\ell \leq k$, the complete outcome of any ℓ-simple (k, n)-election cannot contain b more times than the complete outcome of the k-simple (k, n)-election.

Theorem 21. *Any rule that satisfies strong near-unanimity and weak incentive for minorities can be manipulated by free-riding. If it additionally satisfies issue-wise unanimity and strong monotonicity, then it can be manipulated by free-riding in one issue.*

Proof. Consider a rule that satisfies strong near-unanimity and weak incentive for minorities. Consider any $n \geq 3$ and let k be the number of issues whose existence is prescribed by weak incentive for minorities. Focus on the k-simple (k, n)-election $\mathcal{E}$ and let I be the set of issues where b is the outcome for $\mathcal{E}$. Consider now a $|I|$-simple (k, n)-election $\mathcal{E}'$ where voter n approves of b in every issue in I. By strong near-unanimity, the outcome here must contain b at most $|I|-1$ times. Let i be some issue where in $\mathcal{E}'$ voter n approves of b but a is in the outcome. But then voter n can free-ride in $\mathcal{E}'$ via $\mathcal{E}$ in issues $[k] \setminus I$.

Next, suppose the rule additionally satisfies issue-wise unanimity and strong monotonicity. Again, fix $n \geq 3$ and let k be the number of issues whose existence is prescribed by weak incentive for minorities. Let s be the number of times where b wins in the k-simple (k, n)-election. We know by weak incentive for minorities that $1 \leq s \leq k-1$. Let ℓ be the minimal ℓ for which there is an ℓ-simple (k, n)-election where b wins exactly s times, and let $\mathcal{E}$ be such an election. Observe that $\ell > s$: indeed, we get $\ell \neq s$ via strong near-unanimity and $\ell \geq s$ via issue-wise unanimity. Furthermore, for all $(\ell-1)$-simple (k, n)-elections, b wins at most $s-1$ times (by strong monotonicity). Let i be some issue where b wins in $\mathcal{E}$; note that in this issue voter n must approve of b (by issue-wise unanimity). Next, let j be some issue distinct from i where voter n approves of b and where a is in the outcome (such a j exists as $\ell > s$). Let $\mathcal{E}'$ be the $(\ell-1)$-simple (k, n)-election obtained by letting voter n approve of a instead of b in issue j. Observe that voter n can free-ride in $\mathcal{E}'$ via $\mathcal{E}$ in issue j, completing the proof. □

B Omitted Proofs: Computational Complexity of Free-Riding

Theorem 10. *Successful-$\mathcal{R}$-Free-Riding is* NP*-complete for the sequential egalitarian rule.*

Proof. First, note that, for the egalitarian rule, Successful-$\mathcal{R}$-Free-Riding is in NP. Indeed, given an insincere approval ballot for the manipulator, we can check whether it improves her satisfaction in polynomial time and whether it is a case of free-riding.

Now, we show hardness by a reduction from 3-SAT. Let ϕ be a 3-CNF with n variables and m clauses. We refer to the j-th clause as C_j. We assume w.l.o.g. that ϕ is not satisfied by setting all variables to false and that each clause contains exactly three literals. We construct an instance of Successful-$\mathcal{R}$-Free-Riding with $2(n+1)$ voters and $5n+1$ rounds. In particular, we will have two voters v_i and $\bar{v}_i$ for each variable x_i, a voter u, and a distinguished voter v, the manipulator.

Let us start with the first $4n$ rounds. We subdivide this set of rounds into quadruples; that is, the first quadruple consists of rounds 1, 2, 3 and 4, the second are the rounds 5, 6, 7 and 8, and so on. We refer to the j-th round of quadruple i as round (i, j); for example, round $(2, 3)$ corresponds to round 7. In each round of every quadruple, there are two candidates c and $\bar{c}$. Here (and in all subsequent rounds), we assume that if there is a tie c loses.

Consider a generic quadruple i. In round $(i, 1)$, all voters vote for $\bar{c}$. In round $(i, 2)$, voters v and $\bar{v}_i$ vote for c, while voters u and v_i vote for $\bar{c}$. Everyone else approves of both. Furthermore, in round $(i, 3)$, voters v and u approve of c and $\bar{c}$, respectively; everyone else approves of both. Finally, in round $(i, 4)$, voter v approves of both c and $\bar{c}$, while everyone else approves only of c.

For each quadruple i, we claim that *(i)* v can free-ride (only) in round $(i, 1)$ *(ii)* if v does not free-ride, the winners in this quadruple are $(\bar{c}, \bar{c}, c, c)$ and *(iii)* if v free-rides, the winners in this quadruple are $(\bar{c}, c, \bar{c}, \bar{c})$. We show so by induction over the quadruples.

Consider quadruple 1. Observe that, by tiebreaking, $\bar{c}$ wins in round $(1, 1)$ (irrespectively of what v votes for). Hence, here v can free-ride. Suppose that she votes truthfully. Thus, in the next round, the minimal satisfaction if c or $\bar{c}$ win is the same (namely, 1). By tiebreaking, $\bar{c}$ wins. Now, up to here, every voter has satisfaction 2, save for v and $\bar{v}_1$, who have satisfaction 1. In the next round, then, the minimal satisfaction if c or $\bar{c}$ win is 2 and 1, respectively; hence, c wins. Finally, in round $(1, 4)$, the minimal satisfaction of c winning is 3, whereas the minimal satisfaction in case $\bar{c}$ wins is 2 (namely, of voter u); hence, c wins. With a similar line of reasoning, one can show that $(\bar{c}, c, \bar{c}, \bar{c})$ is the result if v does free-ride. Observe that v can indeed free-ride only in round $(1, 1)$: in every other round, either she is losing, or her vote would change the outcome.

Now, suppose this property holds up to quadruple i, and consider quadruple $i+1$. If this holds, observe that no voter v_j or $\bar{v}_j$ can have won fewer rounds than v or u, and v and u won the same number of rounds. Therefore, at the beginning of each quadruple, v and u are among the voters with the lowest satisfaction. Let this minimal satisfaction be s.

Again, observe that v can free-ride in round $(i+1,1)$. Suppose she votes truthfully. Then, she and u will have the same satisfaction of $s+1$ in round $(i+1,2)$, and $\bar{c}$ will win by tiebreaking. Next, in round $(i+1,3)$, v will have the minimal satisfaction of $s+1$, and hence c will win. Finally, in round $(i+1,4)$, if c wins the minimal satisfaction will be $s+3$, whereas if $\bar{c}$ wins it will be $s+2$ (namely, of u); hence c wins. With similar arguments, we could show that $(\bar{c}, c, \bar{c}, \bar{c})$ is the result if v does free-ride. Observe that v can indeed free-ride only in round $(i+1,1)$: in every other round, either she is losing, or her vote would change the outcome.

Now, let us consider round $4n+1$ to round $5n-1$. From the previous discussion we know that, in each quadruple i, all voters v_j and $\bar{v}_j$ (with $j \neq i$) win the same amount of rounds (either 3 or 4, depending on whether v free-rides or not). Let this number be ℓ_i. Furthermore, one voter in v_i and $\bar{v}_i$ wins ℓ_i rounds, while the other wins $\ell_i - 1$ rounds. Finally, both v and u won exactly $\ell_i - 1$ rounds. Thus, we can partition the voters v_i and $\bar{v}_i$ into two groups, with a satisfaction differing of exactly 1 point. Let s be the satisfaction of the voters in the group with the lowest satisfaction. Observe that the satisfaction of v and u will be exactly $s+1-n$, because in each quadruple i, both v and u lose exactly one round (compared to the voters v_j and $\bar{v}_j$ with $i \neq j$). So then, in each of the rounds from $4n+1$ to $5n-1$, there are two candidates: c and $\bar{c}$. Here, v approves of $\bar{c}$, u of both candidates, and everyone else only of c. Observe that v and u win every such round, as v always has a strictly lower satisfaction than the rest of the voters, and u always approves of all candidates. Furthermore, v can't free-ride here: as she always has the lowest satisfaction, she is always pivotal, and hence her vote decides the outcome. Thus, after these rounds, both v and u will have satisfaction s.

In round $5n$, there are two candidates: $\bar{c}$ and c. Here, v votes for $\bar{c}$, whereas every one else votes for c. If either candidate wins, the minimal satisfaction will be s, and thus $\bar{c}$ wins by tiebreaking. Furthermore, v cannot free-ride: if she were to vote for c, then c would win.

Finally, in round $5n+1$, we know that v has satisfaction of $s+1$ and u of s. Furthermore, if v_i has satisfaction s if c won in round $(i,1)$ and $s+1$ otherwise (and similarly for $\bar{v}_i$ and $\bar{c}$). In this round, there are $m+1$ candidates, namely $c, c_1, \ldots, c_m$. Here, u approves of all candidates, voter v_i (resp. $\bar{v}_i$) approves of c and of all candidates c_j such that $x_i \notin C_j$ (resp. $\bar{x}_i \notin C_j$). Finally, voter v approves of c.

Observe that we can interpret c winning in round $(i,2)$ as setting x_i to true (and conversely for $\bar{c}$); we claim that this assignment satisfies ϕ if and only if c wins in this final round. To see this, observe that if c wins the minimal satisfaction will be $s+1$ (all voters approve of c). Now, consider a clause C_j and its candidate c_j. If all literals in C_j are unsatisfied, then the corresponding voters have all (up to this round) satisfaction $s+1$. Hence, if c_j would win, the minimal satisfaction will be at least $s+1$, as all other voters (except v) approve of it, and v has satisfaction at least $s+1$. Hence, c_j would win by tiebreaking. Conversely, if at least one literal in C_j is satisfied, there is at least one voter with satisfaction s that does not vote for c_j: hence, c_j loses against c. Our claim follows.

Now, observe that, if v were to always vote truthfully, her true satisfaction would be $4n$ (she would win three rounds per quadruple, all rounds from $4n+1$ to $5n$, and lose the last round, by the assumption that setting all variables to false does not satisfy ϕ). Observe also that, as we discussed before, she can only free-ride in the first round of every quadruple. Therefore, the only way she can raise her satisfaction to $4n+1$ is by winning the last round (observe that if she free-rides in some quadruple, she still truly wins three rounds). To do so, she has to force a satisfying assignment for ϕ by free-riding. It follows that v can free-ride if and only if ϕ is satisfiable, and so we are done. □

Theorem 12. *Generalized-Successful-$\mathcal{R}$-Free-Riding is* NP*-complete for every sequential α-OWA rule such that, for all n, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is nonincreasing and $\alpha_1 > \alpha_n$.*

Proof. Fix an α-OWA rule $\mathcal{R}$ satisfying the conditions of the theorem. First, notice that Generalized-Successful-$\mathcal{R}$-Free-Riding is in NP, as we can guess an insincere approval ballot for the manipulator and check whether it improves her satisfaction (and is an instance of generalized free-riding) in polynomial time.

Next, we show hardness by a reduction from 3-SAT. Let ϕ be a 3-CNF with n variables and m clauses. We refer to the j-th clause as C_j. We assume w.l.o.g. that ϕ is not satisfied by setting all variables to false and that each clause contains exactly three literals. We will construct an instance of Generalized-Successful-$\mathcal{R}$-Free-Riding with $2n + 5$ voters. More specifically, there are two voters v_i and $\bar{v}_i$ for each variable x_i, four voters $u_1, \dots, u_4$, plus one distinguished voter v who will try to manipulate.

Given the weight vector $\alpha = (\alpha_1, \dots, \alpha_{2n+5})$, we distinguish three (not necessarily exclusive) cases: (1) $\alpha_2 > \alpha_{n+5}$, (2) $\alpha_{n+1} > \alpha_{2n+5}$, and (3) $\alpha_2 = \alpha_{2n+5}$. Since $\alpha_1 > \alpha_{2n+5}$, at least one case must be true. In the following, we will give a different reduction for each of the three cases.

First case: $\alpha_2 > \alpha_{n+5}$. We construct an instance with $n + 2$ rounds as follows. In the first $n + 1$ rounds, there are two candidates: c and $\bar{c}$. In the last round, there are $m + 1$ candidates, namely $c, c_1, \dots, c_m$. We assume that, in the case of ties, c always loses.

In the first round, everybody votes for $\bar{c}$ except for v and u_1, who vote for c. Here, $\bar{c}$ wins by tiebreaking, and v cannot manipulate.

In each round i with $i \in \{2, \dots, n + 1\}$, voter v_i votes for candidate $\bar{c}$, voter $\bar{v}_i$ for candidate c; everyone else votes for both candidates. We claim that v can manipulate in every such round i and force the win of either c or $\bar{c}$. We show so by induction. In round 2, suppose that v votes only for c (the case where she votes for $\bar{c}$ is analogous). Then, if c were to win, the satisfaction vector would be of form $(1, 1, 1, 2, \dots, 2)$ (everyone but v_1 wins). On the other hand, if $\bar{c}$ wins, then it would be of form $(0, 1, 1, 2, \dots, 2)$ (everyone but $\bar{v}_1$ and v win). Hence, c wins. Observe that if v votes truthfully, $\bar{c}$ wins by tiebreaking. Now, suppose this holds up to a round i. Before round $i + 1$, v has won $i - 1$ rounds (all but the first one), whereas voters $u_1, \dots, u_4$, as well as any pair of voters $v_j, \bar{v}_j$ (with $j \geq i$) have won i rounds. Furthermore, for every pair $v_j, \bar{v}_j$ (with $j < i$), exactly one voter won $i - 1$ rounds while the other i rounds. Suppose again that v votes for c (the case where she votes for $\bar{c}$ is similar). Then, observe that, if c or $\bar{c}$ win, the satisfaction vectors (excluding v) would be completely symmetric (and every voter would have at least a score of i); however, if $\bar{c}$ wins, v would have a satisfaction of $i - 1$, whereas if c wins, she would get a satisfaction of i. Hence, since $\alpha_1 > 0$, here c wins. Observe again that if v votes truthfully, then $\bar{c}$ wins.

Consider the final round. Up to here, v and u_1 have won n rounds (they lost the first round), while u_2, u_3, u_4 have won $n + 1$ rounds. Furthermore, every voter v_i has won n rounds if c won in round $i + 1$ and $n + 1$ times otherwise (and conversely for $\bar{v}_i$ and $\bar{c}$). In this round, voters $u_1, \dots, u_4$ approve of all candidates but c, voter v_i (resp. $\bar{v}_i$) approves of c and every candidate c_j such that $x_i \notin C_j$ (resp. $\bar{x}_i \notin C_j$). Finally, voter v approves of c. Observe that we can interpret c winning in round $i + 1$ as setting x_i to true, and $\bar{c}$ winning as setting x_i to false. We claim that c wins in the last round if and only if this assignment satisfies ϕ. To see this, consider that, if c were to win, the satisfaction vector would be:

$$(n, \underbrace{n+1, \dots, n+1}_{n+4 \text{ times}}, \underbrace{n+2, \dots, n+2}_{n \text{ times}}).$$

Let us call this vector s. Consider now a candidate c_j and its corresponding clause C_j. If all three of its literals are unsatisfied, then the corresponding voters all have satisfaction $n + 1$. Hence, if c_j were to win in this case, the satisfaction vector would again be exactly s. By our assumptions on tiebreaking, here c_j would win against c. Furthermore, suppose that either one, two, or three of the literals have been satisfied. Then, the vectors are, respectively:

$$(n, n, \underbrace{n+1, \ldots, n+1}_{n+2 \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n+1 \text{ times}})$$
$$(n, n, n, \underbrace{n+1, \ldots, n+1}_{n \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n+2 \text{ times}})$$
$$(n, n, n, n, \underbrace{n+1, \ldots, n+1}_{n-2 \text{ times}}, \underbrace{n+2, \ldots, n+2}_{n+3 \text{ times}})$$

Let these vectors be s_1, s_2 and s_3, respectively. One can show that if $\alpha_2 > \alpha_{n+5}$ the dot product between α and each of these three vectors would be strictly lower than the dot product between α and s. For example:

$$\begin{aligned} s \cdot \alpha > s_1 \cdot \alpha \iff \alpha_1 n + \left(\sum_{i=2}^{n+5} \alpha_i (n+1)\right) + \left(\sum_{i=n+6}^{2n+5} \alpha_i (n+2)\right) > \\ (\alpha_1 + \alpha_2) n + \left(\sum_{i=3}^{n+4} \alpha_i (n+1)\right) + \left(\sum_{i=n+5}^{2n+5} \alpha_i (n+2)\right) \iff \\ (\alpha_2 + \alpha_{n+5})(n+1) > \alpha_2 n + \alpha_{n+5}(n+2) \iff \alpha_2 > \alpha_{n+5}. \end{aligned}$$

The other two cases are similar. Hence, if C_j is satisfied, candidate c_j cannot win against c. Consequently, if all clauses are satisfied, candidate c wins.

Now, if c wins in the last round, then the satisfaction of v would be $n + 1$; if c loses, it would be n. Notice also that v cannot raise her satisfaction by manipulating in the final round. Furthermore, if v always submits her true preferences, then by tiebreaking $\bar{c}$ would win in every round i with $i \in \{2, \ldots, n + 1\}$. By assumption, this would not satisfy ϕ, and hence c would not win in the last round. Therefore, v has an incentive to manipulate in these rounds to try and choose a satisfying assignment for ϕ. It follows that v can manipulate via generalized free-riding if and only if ϕ is satisfiable, so we are done.

Second case: $\alpha_{n+1} > \alpha_{2n+5}$. We construct an instance with $n + 2$ rounds as follows. In the first $n + 1$ rounds, there are two candidates, c and $\bar{c}$. In the last round, there are $m + 1$ candidates, namely $c, c_1, \ldots, c_m$. We assume that, in the case of ties, c always loses.

In the first round, $v, u_1, \ldots, u_4$ approve of c, whereas everyone else approves of $\bar{c}$.

In each round i with $i \in \{2, \ldots, n + 1\}$, voter v_i votes for candidate c, voter $\bar{v}_i$ for candidate $\bar{c}$, and everyone else votes for both candidates.

In the final round, voters v, u_1, u_2 approve of c, voter v_i (resp. $\bar{v}_i$) approves of candidate c_j if $x_i \in C_j$ (resp. $\bar{x}_i \in C_j$). Finally, voters u_3 and u_4 approve of all candidates.

We claim that *(i)* voters $v, u_1, \ldots, u_4$ lose in the first round, and that in the following n rounds *(ii)* candidate $\bar{c}$ wins if v votes truthfully and *(iii)* v can force the win of either c or $\bar{c}$ by manipulating. The arguments for this are essentially the same as in the first case.

Now, consider the last round. Up to here, $v, u_1, \ldots, u_4$ have won n rounds (all but the first). Furthermore, every voter v_i has won $n+1$ rounds if c won in round i and n times otherwise (and conversely for $\bar{v}_i$ and $\bar{c}$). We interpret again c winning in round i as setting x_i to true, and $\bar{c}$ winning as setting x_i to false. We claim that c wins in the last round if and only if this assignment satisfies ϕ. To see this, consider that, if c were to win, the satisfaction vector would be:

$$(\underbrace{n, \dots, n}_{n \text{ times}}, \underbrace{n+1, \dots, n+1}_{n+5 \text{ times}}).$$

Let us call this vector s. Consider now a candidate c_j and its corresponding clause C_j. If all three of its literals are unsatisfied, then the corresponding voters all have satisfaction n. Hence, if c_j were to win in this case, the satisfaction vector would again be exactly s. By our assumptions on tiebreaking, here c_j would win against c. Furthermore, suppose that either one, two, or three of the literals have been satisfied. Then, the vectors are, respectively:

$$(\underbrace{n, \dots, n}_{n+1 \text{ times}}, \underbrace{n+1, \dots, n+1}_{n+3 \text{ times}}, n+2)$$
$$(\underbrace{n, \dots, n}_{n+2 \text{ times}}, \underbrace{n+1, \dots, n+1}_{n+1 \text{ times}}, n+2, n+2)$$
$$(\underbrace{n, \dots, n}_{n+3 \text{ times}}, \underbrace{n+1, \dots, n+1}_{n-1 \text{ times}}, n+2, n+2, n+2)$$

One can show that if $\alpha_{n+1} > \alpha_{2n+5}$ the dot product between α and each of these three vectors would be strictly lower than the dot product between α and s. The computation is analogous to the one we did in the first case. Hence, if C_j is satisfied, candidate c_j cannot win against c. Consequently, if all clauses are satisfied, candidate c wins. With similar arguments as before, we conclude that here v can manipulate if and only if ϕ is satisfiable.

Third case: $\alpha_2 = \alpha_{2n+5}$. We construct an instance with $n+3$ rounds as follows. The rounds $2, \dots, n+1$ and the last round are equal to the first case. The first round is almost identical, save for the fact that u_1 also votes for $\bar{c}$.

Hence, we know that v loses in the first round and that she can manipulate in all the following rounds to force a win of either c or $\bar{c}$. Let us focus on round $n+2$. We will design this round to make sure that only v wins, and that she cannot manipulate via generalized free-riding. We distinguish two sub-cases:

(1) $\alpha_2 = \dots = \alpha_{2n+5} = 0$. Here, everyone votes for c, save for v, who votes for $\bar{c}$. There are no other candidates. Observe that, in case either c or $\bar{c}$ wins here, the minimal satisfaction will be n in both cases; $\bar{c}$ wins by tiebreaking. Furthermore, were v to vote for c, then c would win (as the minimal satisfaction for c winning would raise to $n+1$).

(2) $\alpha_2 = \dots = \alpha_{2n+5} > 0$. Here, there is one candidate c_{v^*} for every voter $v^* \in N$, and we assume that in case of ties c_v wins. Furthermore, we assume that all voters vote for their voter-candidate. We show that here all candidates receive the same score. Consider any two candidates c_y and c_z. Let $y = (y_1, \dots, y_{2n+5})$ and $z = (z_1, \dots, z_{2n+5})$ be the satisfaction vectors corresponding to c_y and c_z winning, respectively. For both c_y and c_z, there is surely at least one voter with satisfaction n that disapproves of them; hence, $y_1 = z_1 = n$. Furthermore, there is exactly one voter approving each candidate, and hence $\sum_{i=2}^{2n+5} y_i = \sum_{i=2}^{2n+5} z_i$. These two facts, together with the fact that $\alpha_2 = \dots = \alpha_{2n+5}$, imply that $\alpha \cdot y = \alpha \cdot z$. Hence, every two candidates receive the same score: by tiebreaking, c_v wins. Now, notice that, if v approves of any other candidate c_{v^*} distinct from c_v, then c_{v^*} will receive a strictly greater score than any other candidate (as now two voters approve of it).

Now, consider the last round. Up to here, $v, u_1, \dots, u_4$ won $n+1$ rounds. Furthermore, every voter v_i has won $n+1$ rounds if c won in round i and n times otherwise (and conversely for $\bar{v}_i$ and $\bar{c}$). We interpret again c winning in round i as setting x_i to true, and $\bar{c}$ winning as setting x_i to false. We claim that c wins in the last round if and only if this assignment satisfies ϕ. To see this, consider that, if c were to win, the satisfaction vector would be:

$$(\underbrace{n+1,\ldots,n+1}_{n+4 \text{ times}},\underbrace{n+2,\ldots,n+2}_{n+1 \text{ times}}).$$

Let us call this vector s. Consider now a candidate c_j and its corresponding clause C_j. If all three of its literals are unsatisfied, then the corresponding voters all have satisfaction $n+1$. Hence, if c_j were to win in this case, the satisfaction vector would again be exactly s. By our assumptions on tiebreaking, here c_j would win against c. Furthermore, suppose that either one, two, or three of the literals have been satisfied. Then, the vectors are, respectively:

$$(n,\underbrace{n+1,\ldots,n+1}_{n+2 \text{ times}},\underbrace{n+2,\ldots,n+2}_{n+2 \text{ times}})$$
$$(n,n,\underbrace{n+1,\ldots,n+1}_{n \text{ times}},\underbrace{n+2,\ldots,n+2}_{n+3 \text{ times}})$$
$$(n,n,n,\underbrace{n+1,\ldots,n+1}_{n-2 \text{ times}},\underbrace{n+2,\ldots,n+2}_{n+4 \text{ times}})$$

One can show that if $\alpha_1 > \alpha_{n+4}$ (which is implied by $\alpha_1 > \alpha_{2n+5}$ and $\alpha_2 = \alpha_4 = \alpha_{2n+5}$) the dot product between α and each of these three vectors would be strictly lower than the dot product between α and s. The computation is analogous to the one we did in the first case. Hence, if C_j is satisfied, candidate c_j cannot win against c. Consequently, if all clauses are satisfied, candidate c wins. With similar arguments as the first case, we conclude that here v can manipulate if and only if ϕ is satisfiable.

This concludes the proof. □

C Omitted Proofs: Free-Riding with Optimization-Based Rules

Theorem 13. *Every opt-Thiele and opt-OWA rule except the utilitarian rule can be manipulated by free-riding.*

Proof. Let $\mathcal{R}$ be an opt-f-Thiele Rule different from the utilitarian rule. Then, there exists a k such that $f(k-1) > f(k)$. Consider a $k+1$ issue election with four voters and two candidates a and b such that for the first k issues all voters only approve candidate a. Moreover, on issue $k+1$ voters 1 and 2 approve b while voter 3 and 4 approve a. Assume further that a is preferred to b in the tiebreaking order. Clearly, selecting b in one of the first k rounds just reduces the score of all voters, hence in any optimal outcome a wins in the first k issues. Letting a or b win on issue $k+1$ increases the score of the outcome by $2f(k)$ for both candidates. We can assume that a wins by tiebreaking. We claim that voter 1 can manipulate by changing her vote in one of the first k issues to $\{b\}$. Assume that 1 manipulates on issue k. Then, it is still clearly best to let a win in the first $k-1$ issues. This leads to score of $S := 4\sum_{i=1}^{k-1} f(i)$. Let us now consider the score of four possible outcomes on issue k and $k+1$. The outcome $(a,\ldots,a,a)$ has score of $S + 3f(k-1) + 2f(k)$, the outcome $(a,\ldots,a,b)$ has score of $S + 3f(k-1) + f(k) + f(k-1)$, the outcome $(a,\ldots,b,a)$ has score of $S + f(k-1) + 2f(k-1)$ and the outcome $(a,\ldots,b,b)$ has score of $S + f(k-1) + f(k) + f(k-1)$. As, $f(i-1) > f(k)$ this implies that $(a,\ldots,a,b)$ is the winning outcome. Therefore, voter 1 did free-ride successfully.

Now, let $\mathcal{R}$ be an opt-OWA rule that is not the utilitarian rule. Then there exists a n for which the vector α for k voters satisfies $\alpha_1 > \alpha_n$. Clearly, $n \geq 2$. Consider an election with 2 issues and n voters. In each issue there are n candidates $a_1,\ldots a_n$. In the first issue, voters 1 and 2 approve a_1. Every other voter $i \in \{3,\ldots,n\}$ approves a_i. In the second issue voter 1 approves a_1, voter 2 approves a_2 and all other voters approve both a_1 and a_2. We assume that candidates with a lower index are preferred by the tiebreaking, which is applied lexicographically. Selecting a candidate other than a_1 in the first issue leads to satisfaction vector $(0, 1, \ldots, 1, 2)$, independently of whether a_1 or a_2 is selected in issue 2. On the other hand, selecting a_1 in issue 1 leads to satisfaction vector $(1, 1, \ldots, 1, 2)$ independently of whether a_1 or a_2 is selected in issue 2. This means (a_1, a_1) and (a_1, a_2) lead to the highest OWA score. By tiebreaking, (a_1, a_1) wins. Now, we claim that voter 2 can free-ride by approving a_2 instead of a_1 in the first issue. Assume first, that a candidate other than a_1 or a_2 is selected in the first issue. This still leads to a satisfaction vector of $(0, 1, \ldots, 1, , 2)$, independently of whether a_1 or a_2 is selected in issue 2. Choosing a_1 in both issues leads to a satisfaction vector of $(0, 1, \ldots, 1, 2)$. Choosing a_1 in issue 1 and a_2 in issue 2 leads to vector $(1, \ldots, 1)$. Choosing a_2 both times or first a_2 and then a_1 is symmetric. As $\alpha_1 > \alpha_n$ we know that

$$\alpha \cdot (\underbrace{1, \ldots, 1}_{n \text{ times}}) = \sum_{i=1}^{n} \alpha_i > \alpha_n - \alpha_1 + \sum_{i=1}^{n} \alpha_i = \alpha \cdot (0, \underbrace{1, \ldots, 1}_{n-2 \text{ times}}, 2).$$

It follows that (a_1, a_2) and (a_2, a_1) are the outcomes maximizing the OWA score. By tiebreaking, (a_1, a_2) is the winning outcome. It follows that 2 did successfully free-ride. □

Proposition 14. *Free-riding cannot reduce the satisfaction of the free-riding voter when an opt-Thiele rule is used, but it can increase the satisfaction of the free-riding voter.*

Proof. Fix the opt-f-Thiele rule. It follows directly from Theorem 13 that free-riding can increase the satisfaction of the free-riding voter. Let us show that it can never decrease the satisfaction of the free-riding voter. Let $\mathcal{E}$ be an election, $\bar{w}$ be the outcome of $\mathcal{E}$ under the f-Thiele rule, and consider a voter v^* such that v^* can free-ride in issues $I \subset [k]$. Finally, let $\mathcal{E}^*$ be the election after v^* free-rides and $\bar{w}^*$ the outcome of $\mathcal{E}^*$ under the same f-Thiele rule. Now, as v^* free-rides, i.e., the winners in issues I are the same in $\bar{w}$ and $\bar{w}^*$, we know that v^* approves the winners of issues I in her honest ballot in $\mathcal{E}$ and does not approve the winners of issues I in her manipulated ballot in $\mathcal{E}^*$. It follows the satisfaction of v^* with $\bar{w}^*$ resp. $\bar{w}$ in $\mathcal{E}$ is higher by one than in $\mathcal{E}^*$, i.e.,

$$\begin{aligned} sat_{\mathcal{E}^*}(v^*, \bar{w}^*) &= sat_{\mathcal{E}}(v^*, \bar{w}^*) - |I| \quad \text{as well as} \\ sat_{\mathcal{E}^*}(v^*, \bar{w}) &= sat_{\mathcal{E}}(v^*, \bar{w}) - |I|. \end{aligned} \tag{1}$$

All other voters submit the same ballot in $\mathcal{E}$ and $\mathcal{E}^*$. Hence, for all $v \neq v^*$ we have

$$\begin{aligned} sat_{\mathcal{E}^*}(v, \bar{w}^*) &= sat_{\mathcal{E}}(v, \bar{w}^*) \quad \text{as well as} \\ sat_{\mathcal{E}^*}(v, \bar{w}) &= sat_{\mathcal{E}}(v, \bar{w}). \end{aligned} \tag{2}$$

Now assume for the sake of a contradiction that $sat_{\mathcal{E}}(v^*, \bar{w}) > sat_{\mathcal{E}}(v^*, \bar{w}^*)$, i.e., free-riding led to a lower satisfaction for v^* with respect to her honest ballot.

As $\bar{w}^*$ is the winning outcome of $\mathcal{E}^*$, we know that

$$\sum_{v \in N} \sum_{i=1}^{sat_{\mathcal{E}^*}(v, \bar{w}^*)} f(i) \quad > \quad \sum_{v \in N} \sum_{i=1}^{sat_{\mathcal{E}^*}(v, \bar{w})} f(i).$$

Note that the inequality is strict because, if the two sides were equal, it would mean that w^* is preferred over w by the tiebreaking and that, by Equations (1) and (2), that w and w^* would have the same score also in election $\mathcal{E}$. Hence, w^* would win in $\mathcal{E}$, contradicting our assumptions. Next, by the previous inequality and Equations (1) and (2),

$$\sum_{i=1}^{sat_{\mathcal{E}}(v^*, \bar{w}^*) - |I|} f(i) + \sum_{v \in N \setminus \{v^*\}} \sum_{i=1}^{sat_{\mathcal{E}}(v, \bar{w}^*)} f(i) \quad > \quad \sum_{i=1}^{sat_{\mathcal{E}}(v^*, \bar{w}) - |I|} f(i) + \sum_{v \in N \setminus \{v^*\}} \sum_{i=1}^{sat_{\mathcal{E}}(v, \bar{w})} f(i).$$

This can be rewritten as

$$\sum_{v\in N}\sum_{i=1}^{sat_{\mathcal{E}}(v,\bar{w}^*)} f(i) \quad - \sum_{i=sat_{\mathcal{E}}(v^*,\bar{w}^*)-|I|+1}^{sat_{\mathcal{E}}(v^*,\bar{w}^*)} f(i) \quad > \quad \sum_{v\in N}\sum_{i=1}^{sat_{\mathcal{E}}(v,\bar{w})} f(i) \quad - \sum_{i=sat_{\mathcal{E}}(v^*,\bar{w})-|I|+1}^{sat_{\mathcal{E}}(v^*,\bar{w})} f(i),$$

which gives

$$\sum_{v\in N}\sum_{i=1}^{sat_{\mathcal{E}}(v,\bar{w}^*)} f(i) - \sum_{v\in N}\sum_{i=1}^{sat_{\mathcal{E}}(v,\bar{w})} f(i) \quad > \sum_{i=sat_{\mathcal{E}}(v^*,\bar{w}^*)-|I|+1}^{sat_{\mathcal{E}}(v^*,\bar{w}^*)} f(i) \quad - \sum_{i=sat_{\mathcal{E}}(v^*,\bar{w})-|I|+1}^{sat_{\mathcal{E}}(v^*,\bar{w})} f(i). \tag{3}$$

Next, we know that the left-hand side of (3) must be non-positive, as $\bar{w}$, and not $\bar{w}^*$, is the winner of $\mathcal{E}$. Moreover, we assumed that $sat_{\mathcal{E}}(v^*,\bar{w}) > sat_{\mathcal{E}}(v^*,\bar{w}^*)$, which together with the non-increasingness of f implies that the right-hand side of (3) must be non-negative: a contradiction. Hence, we get $sat_{\mathcal{E}}(v^*,\bar{w}) \leq sat_{\mathcal{E}}(v^*,\bar{w}^*)$, completing the proof. □

Proposition 15. *Free-riding cannot reduce the satisfaction of the free-riding voter when opt-leximin is used, but it can increase the satisfaction of the free-riding voter.*

Proof. It follows directly from Theorem 5 that free-riding can increase the satisfaction of the free-riding voter. Let us show that it can never decrease the satisfaction of the free-riding voter. Let $\mathcal{E}$ be an election, $\bar{w}$ be the outcome of $\mathcal{E}$ under the leximin rule, and consider a voter v^* such that v^* can free-ride in issues $I \subset [k]$. Finally, let $\mathcal{E}^*$ be the election after v^* free-rides and $\bar{w}^*$ the outcome of $\mathcal{E}^*$ under opt-leximin. In the following we write $N_i^{\mathcal{E}}(\bar{w}) = \{v \in N \mid sat_{\mathcal{E}}(v,\bar{w}) = i\}$. Now, as v^* free-rides, i.e., the winners in issues I are the same in $\bar{w}$ and $\bar{w}^*$, we know that v^* approves the winners of issues I in her honest ballot in $\mathcal{E}$ and does not approve the winners of issues I in her free-riding ballot in $\mathcal{E}^*$. It follows the satisfaction of v^* with $\bar{w}^*$ resp. $\bar{w}$ in $\mathcal{E}$ is higher by one than in $\mathcal{E}^*$, i.e.,

$$\begin{aligned} sat_{\mathcal{E}^*}(v^*,\bar{w}^*) &= sat_{\mathcal{E}}(v^*,\bar{w}^*) - |I| \quad \text{as well as} \\ sat_{\mathcal{E}^*}(v^*,\bar{w}) &= sat_{\mathcal{E}}(v^*,\bar{w}) - |I|. \end{aligned} \tag{4}$$

All other voters submit the same ballot in $\mathcal{E}$ and $\mathcal{E}^*$. Hence, for all $v \neq v^*$ we have

$$\begin{aligned} sat_{\mathcal{E}^*}(v,\bar{w}^*) &= sat_{\mathcal{E}}(v,\bar{w}^*) \quad \text{as well as} \\ sat_{\mathcal{E}^*}(v,\bar{w}) &= sat_{\mathcal{E}}(v,\bar{w}). \end{aligned} \tag{5}$$

Now assume for the sake of a contradiction that $sat_{\mathcal{E}}(v^*,\bar{w}) > sat_{\mathcal{E}}(v^*,\bar{w}^*)$, i.e., free-riding led to a lower satisfaction for v^* with respect to her honest ballot.

First of all, observe that $\bar{w}^*$ and $\bar{w}$ cannot tie in the leximin order in $\mathcal{E}^*$ (i.e., the victory of $\bar{w}^*$ over $\bar{w}$ cannot be due to tiebreaking). Were this not the case, then we would have $|N_j^{\mathcal{E}^*}(\bar{w}^*)| = |N_j^{\mathcal{E}^*}(\bar{w})|$ for every index j. However, $sat_{\mathcal{E}}(v^*,\bar{w}) > sat_{\mathcal{E}}(v^*,\bar{w}^*)$ and (4) imply that $sat_{\mathcal{E}^*}(v^*,\bar{w}) > sat_{\mathcal{E}^*}(v^*,\bar{w}^*)$. This, together with (5), implies that $|N_j^{\mathcal{E}^*}(\bar{w}^*)| \neq |N_j^{\mathcal{E}^*}(\bar{w})|$ for some index j.

Consequently, we know that $\bar{w}^* \succ \bar{w}$ according to the leximin order in $\mathcal{E}^*$. In other words, there is a j such that $|N_j^{\mathcal{E}^*}(\bar{w}^*)| < |N_j^{\mathcal{E}^*}(\bar{w})|$ and $|N_\ell^{\mathcal{E}^*}(\bar{w}^*)| = |N_\ell^{\mathcal{E}^*}(\bar{w})|$ for all $\ell < j$. We claim that the deciding index j cannot be smaller than $sat_{\mathcal{E}^*}(v^*,\bar{w}^*)$ as for all smaller indices $\ell < sat_{\mathcal{E}^*}(v^*,\bar{w}^*)$ it follows from $sat_{\mathcal{E}}(v^*,\bar{w}) > sat_{\mathcal{E}}(v^*,\bar{w}^*)$ that v^* is not in $N_\ell^{\mathcal{E}}(\bar{w}^*)$, $N_\ell^{\mathcal{E}^*}(\bar{w}^*)$, $N_\ell^{\mathcal{E}}(\bar{w})$ and $N_\ell^{\mathcal{E}^*}(\bar{w})$. Therefore, it follows from (5) that $|N_\ell^{\mathcal{E}}(\bar{w}^*)| = |N_\ell^{\mathcal{E}^*}(\bar{w}^*)|$ and $|N_\ell^{\mathcal{E}}(\bar{w})| = |N_\ell^{\mathcal{E}^*}(\bar{w})|$. Hence, $j < sat_{\mathcal{E}^*}(v^*,\bar{w}^*)$ would be a contradiction to the assumption that $\bar{w}$ is the leximin outcome of $\mathcal{E}$ and hence leximin preferred to $\bar{w}^*$ in $\mathcal{E}$.

Therefore, we know that $|N_\ell^{\mathcal{E}^*}(\bar{w}^*)| = |N_\ell^{\mathcal{E}^*}(\bar{w})|$ for all $\ell < sat_{\mathcal{E}^*}(v^*,\bar{w}^*) \leq j$ and

$$|N^{\mathcal{E}^*}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w}^*)| \leq |N^{\mathcal{E}^*}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w})|.$$

It follows that also $|N_\ell^{\mathcal{E}}(\bar{w}^*)| = |N_\ell^{\mathcal{E}^*}(\bar{w}^*)| = |N_\ell^{\mathcal{E}^*}(\bar{w})| = |N_\ell^{\mathcal{E}}(\bar{w})|$ for all $\ell < sat_{\mathcal{E}^*}(v^*, \bar{w}^*) \leq j$. Finally, it follows from (4) that v^* is in $N^{\mathcal{E}^*}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w}^*)$ but not in $N^{\mathcal{E}}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w}^*)$. Moreover, because we assumed $sat_{\mathcal{E}}(v^*, \bar{w}) > sat_{\mathcal{E}}(v^*, \bar{w}^*)$, v^* is neither in $N^{\mathcal{E}^*}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w})$ nor in $N^{\mathcal{E}}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w})$. Therefore, we have

$$|N^{\mathcal{E}}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w}^*)| + 1 = |N^{\mathcal{E}^*}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w}^*)| \leq |N^{\mathcal{E}^*}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w})| = |N^{\mathcal{E}}_{sat_{\mathcal{E}^*}(v^*,\bar{w}^*)}(\bar{w})|.$$

However, that means that $\bar{w}^*$ is leximin preferred to $\bar{w}$ in $\mathcal{E}$, which is a contradiction to the assumption that $\bar{w}$ is the outcome of $\mathcal{E}$. □

Theorem 16. *$\mathcal{R}$-Outcome-Determination is* NP*-hard for every opt-f-Thiele rule distinct from the utilitarian rule.*

Proof. Fix an opt-f-Thiele rule $\mathcal{R}$ distinct from the utilitarian rule. We show hardness by a reduction from CubicVertexCover, a variant of VertexCover where every node has a degree of exactly three (Alimonti and Kann 2000). In the following, let ℓ be the smallest ℓ such that $f(\ell) > f(\ell + 1)$ (such an ℓ must exist by virtue of $\mathcal{R}$ not being the utilitarian rule).

Consider an instance (G, k) of CubicVertexCover. Here, k is a natural number and $G = (V, E)$ is an undirected graph with n nodes and m edges where every node has a degree of 3. We assume w.l.o.g. that $k < m$. We will construct an instance $(\mathcal{E}, \ell + k, c_{d_1})$ of $\mathcal{R}$-Outcome-Determination, where $\mathcal{E} = \langle N, \bar{A}, \bar{C}\rangle$ is an election with $(\ell + k)$ issues and $2m$ voters. Observe in particular that ℓ does not depend on (G, k).

We construct the instance as follows. We have one voter v_e for every edge $e \in E$, plus m extra dummy voters $\{d_1, \ldots, d_m\}$. In the first $\ell - 1$ issues, there are two candidates c and c', and all voters approve of both. In the next k issues, there is one candidate c_η for every node $\eta \in V$, plus one candidate c_{d_i} for every dummy voter d_i. In all of these issues, every edge voter v_e approves of the two candidates c_η and $c_{\eta'}$ such that $e = \{\eta, \eta'\}$. Furthermore, every dummy candidate d_i approves of only c_{d_i}. Finally, in the last issue, there is one candidate c_v for every voter $v \in N$, and every such voter only approves of c_v.

To deal with ties, we assume that each issue i is associated with a total ordering $\succ_i$ such that:

(1) If $i \in \{\ell, \ldots, \ell + k - 1\}$, then node-candidates are preferred over other candidates, and $c_{d_n} \succ_i \cdots \succ_i c_{d_1}$;
(2) If $i = \ell + k$, then all candidates c_{v_e} (with $e \in E$) are preferred over other candidates, and $c_{d_1} \succ_i \cdots \succ_i c_{d_n}$;

We compare outcomes $\bar{w}$ and $\bar{w}'$ with $\bar{w} = (w_1, \ldots, w_{\ell+k})$ and $\bar{w}' = (w'_1, \ldots, w'_{\ell+k})$ lexicographically, that is $\bar{w} \succ \bar{w}'$ if there exists an index $j \in [\ell + k]$ such that $w_1 = w'_1, \ldots, w_{j-1} = w'_{j-1}$ and $w_j > w'_j$. Among the outcomes with maximal scores, we return the maximal outcome according to $\succ$.

We want to show that (G, k) is a yes-instance if and only if $(\mathcal{E}, \ell + k, c_{d_1})$ is. First, note that all voters have a satisfaction of at least $\ell - 1$ (because of the first $\ell - 1$ issues). Next, let us show the following, useful claim:

Claim 1. *Let $\bar{w}$ be an outcome of the election $\mathcal{E}$, let $\mathcal{E}_{[-1]}$ be the election that only differs from $\mathcal{E}$ in that issue $\ell + k$ is missing and let $\bar{w}_{[-1]}$ be $\bar{w}$ restricted to $\mathcal{E}_{[-1]}$. Then*

$$\mathit{Thiele}_f(\bar{w}) = \mathit{Thiele}_f(\bar{w}_{[-1]}) + f(\ell).$$

Proof: First, note that at most one dummy voter can win in each issue in $\{\ell, \ldots, \ell + k - 1\}$. As there are m dummy voters and $k < m$, at least one voter will win no issue in $\{\ell, \ldots, \ell + k - 1\}$. Thus, whatever outcome we fix for issue 1 to $\ell + k - 1$, there will be at least one voter with satisfaction $\ell - 1$. Now, in issue $\ell + k$ every outcome increases the satisfaction of one voter by one. As $f(\ell) > f(\ell + 1) \geq f(\ell^*)$ for $\ell^* \geq \ell + 1$, it is always optimal to pick a candidate corresponding to a voter with satisfaction $\ell - 1$ in $\mathcal{E}_{[-1]}$. ⋄

Using this fact, we show that c_{d_1} wins in the last issue if and only if the candidates selected in issues ℓ to $\ell + k - 1$ correspond to a vertex cover of G.

Let $\bar{w}$ be the winning outcome and assume that the winners of issue ℓ to $\ell + k - 1$ correspond to a vertex cover of G, i.e., that $V[\bar{w}] := \{\eta \in V \mid \exists i \leq k - 1 \text{ s.t. } w_{\ell+i} = c_\eta\}$ is a vertex cover. Then, clearly, every edge voter v_e has a satisfaction of at least ℓ in $\mathcal{E}_{[-1]}$. As we observed above, this means no candidate c_{v_e} can be winning in the last issue. Moreover, c_{d_i} does not win in issues ℓ to $\ell + k - 1$: choosing this candidate cannot give a higher score than choosing another dummy candidate c_{d_i} (with $i > 1$) for a voter d_i with satisfaction $\ell - 1$, as every such candidate is approved by exactly one dummy voter. Moreover, the outcome where we replace c_{d_1} by c_{d_i} will always be preferred by our tiebreaking. Therefore, d_1 has satisfaction $\ell - 1$ in $\mathcal{E}_{[-1]}$. It follows that c_{d_1} must win in the last issue: Selecting a candidate c_{v_e} leads to a worse score, and selecting a candidate c_{d_i} for $i > 1$ does not lead to a higher score but to an outcome that is less preferred by the tiebreaking.

Now assume that the winners of issues ℓ to $\ell + k - 1$ do not correspond to a vertex cover of G. Then there is one edge voter v_e with satisfaction $\ell - 1$ in $\mathcal{E}_{[-1]}$. Hence, selecting c_{d_1} in the last issue cannot lead to a higher score than selecting c_{v_e}, and the latter is preferred lexicographically. Thus, c_{d_1} does not win in the last issue.

It remains to show that if there is a vertex cover of G with at most k vertices, then $V[\bar{w}]$ is a vertex cover for the winning outcome $\bar{w}$. Assume a vertex cover of G of size at most k exists. Further, assume for the sake of a contradiction that $V[\bar{w}]$ is not a vertex cover.

If $V[\bar{w}]$ is not a vertex cover, then for every issue $i \in \{\ell, \dots, \ell + k - 1\}$ the winner must be a vertex candidate c_η. Assume otherwise that there is an issue $i \in \{\ell, \dots, \ell + k - 1\}$ where a candidate c_{d_j} wins. We observe that because $\bar{w}$ does not correspond to a vertex cover, there is at least one voter v_e that has satisfaction $\ell - 1$ in $\mathcal{E}_{[-1]}$. Then w_i contributes at most $f(\ell)$ to the score of $\bar{w}$. If the winner in the last round is not c_{v_e} we can replace c_{d_j} by c_η which would contribute at least $f(\ell)$ to the score and be preferable by tiebreaking. If the winner in the last round is c_{v_e}, then we can replace c_{d_j} in issue i by c_η and c_{v_e} in the last issue by any other candidate corresponding to a voter with satisfaction $\ell - 1$ without the last issue. The score of the resulting outcome is at least as good as the score $\bar{w}$ and it is preferred by tiebreaking.

Now, let $E^\ell_{\bar{w}} := \{v_e \in N \mid e \in E \wedge sat_{\mathcal{E}_{[-1]}}(v_e, \bar{w}_{[-1]}) \geq \ell\}$ be the set of edge-voters with satisfaction at least ℓ in $\bar{w}_{[-1]}$. We define $E^{\ell+1}_{\bar{w}} \subseteq E^\ell_{\bar{w}}$ analogously. Now, we observe that

$$\sum_{v_e \in E^\ell_{\bar{w}}} \Big(sat_{\mathcal{E}_{[-1]}}(v_e, \bar{w}_{[-1]}) - (\ell - 1)\Big) =$$

$$|E^\ell_{\bar{w}}| + \sum_{v_e \in E^{\ell+1}_{\bar{w}}} \Big(sat_{\mathcal{E}_{[-1]}}(v_e, \bar{w}_{[-1]}) - \ell\Big) = 3|V[\bar{w}]| = 3k$$

because the set contains k nodes and each node has degree 3.

Finally, any outcome $\bar{w}^*$ on $\mathcal{E}_{[-1]}$ in which on every issue in $\{\ell, \dots, \ell + k - 1\}$ a vertex candidate c_η wins has the following Thiele score:

$$\sum_{v \in V} \left(\sum_{i=1}^{\ell-1} f(i)\right) + |E^\ell_{\bar{w}}| f(\ell) + \underbrace{\sum_{v_e \in E^{\ell+1}_{\bar{w}}} \sum_{i=\ell+1}^{sat_{\mathcal{E}_{[-1]}}(v_e, \bar{w}_{[-1]})} f(i)}_{3k - |E^\ell_{\bar{w}}| \text{ addends}}.$$

As $f(\ell) > f(\ell + 1)$, this function is maximized by maximizing $|E^\ell_{\bar{w}}|$. As a vertex cover exists, we know that we can reach $|E^\ell_{\bar{w}}| = m$. Hence, the outcome maximizing the Thiele score must do so, which means that it must be a vertex cover. □

Theorem 17. *$\mathcal{R}$-Outcome-Determination is* NP*-hard for every opt-α-OWA rule such that, for all n, α^n is nonincreasing and $\alpha_1 > \alpha_n$.*

Proof. Fix a rule $\mathcal{R}$ satisfying the condition of the theorem. We show hardness by a reduction from CubicVertexCover. Consider an instance (G, k) of this problem. Here, $G = (V, E)$ is a graph with n nodes and m edges where each node has a degree of exactly three, and $k \in \mathbb{N}$. We assume w.l.o.g. that $k < n$. We construct an instance of $\mathcal{R}$-Outcome-Determination with $(k+1)$ issues and $3m$ voters. As $\alpha_1 > \alpha_{3m}$, there are two cases:

(1) There is a $p \in [2m]$ such that $\alpha_p > \alpha_{p+1}$, or
(2) There is a $p > 2m$ with $p < 3m$ such that $\alpha_1 = \alpha_p > \alpha_{p+1}$.

In the following, we treat these cases separately.

First case. We construct an instance $(\mathcal{E}, k+1, c_{d_1})$ of $\mathcal{R}$-Outcome-Determination. Here, we have one voter v_e for each edge $e \in E$, and two sets of dummy voters, $\{d_1, \dots, d_p\}$ and $\{w_1, \dots, w_{2m-p}\}$. In the first k issues, there is one candidate c_η for each node $\eta \in V$, plus one dummy candidate c_{d_i} for each dummy voter d_i. Here, each edge-voter v_e approves of the two node-candidates v_η and $v_{\eta'}$ such that $e = \{\eta, \eta'\}$. Moreover, each dummy voter d_i approves only of dummy candidate c_{d_i}, and all dummy candidates w_i approve of all candidates. In the last issue, there is one candidate c_v for all voters $v \in N \setminus \{w_i\}_{i \in [2m-p]}$, and every such v only approves of c_v. Finally, here, all voters in $\{w_i\}_{i \in [2m-p]}$ approve of all candidates.

We use the following tiebreaking mechanism, which is essentially identical to the one used in the proof of Theorem 16. We assume that each issue i is associated with a total ordering $\succ_i$ such that:

(1) If $i \in \{1, \dots, k\}$, then node-candidates are preferred over other candidates, and $c_{d_n} \succ_i \cdots \succ_i c_{d_1}$;
(2) If $i = k+1$, then all candidates c_{v_e} (with $e \in E$) are preferred over other candidates, and $c_{d_1} \succ_i \cdots \succ_i c_{d_n}$;

We compare outcomes $\bar{w}$ and $\bar{w}'$ with $\bar{w} = (w_1, \dots, w_{\ell+k})$ and $\bar{w}' = (w'_1, \dots, w'_{\ell+k})$ lexicographically, that is $\bar{w} \succ \bar{w}'$ if there exists an index $j \in [\ell + k]$ such that $w_1 = w'_1, \dots, w_{j-1} = w'_{j-1}$ and $w_j > w'_j$. Among the outcomes with maximal scores, we return the maximal outcome according to $\succ$.

We want to show that (G, k) is a yes-instance if and only if $(\mathcal{E}, k+1, c_{d_1})$ is. Suppose that there exists a vertex cover for G with size at most k. First, we show that all edge-voters must win at least one issue in $[k]$. Then, we show that, if all edge-voters win at least one issue in $[k]$, then c_{d_1} wins in issue $k+1$.

Let us show that all edge-voters win at least one issue in $[k]$. Let $\bar{w} = \mathcal{R}(\mathcal{E})$, and assume towards a contradiction that some edge-voter v_e never win any issue in $[k]$. Assume that some dummy candidate c_{d_j} wins some issue $i \in [k+1]$. If v_e never wins at all, we can make c_η (for some $\eta \in e$) win in issue i and obtain an outcome that has a greater or equal score (as α is nonincreasing) and is preferred lexicographically. If v_e wins in issue $k+1$, we can make a similar argument by making c_η win issue i and c_{d_j} win issue $k+1$. Hence, in the following, we assume w.l.o.g. that no dummy candidate wins in $\bar{w}$.

Next, let $\bar{w}^*$ be some outcome where all edge-voters win at least one issue in $[k]$ (which is possible, because (G, k) is a yes-instance), no dummy voter wins any issue in $[k]$ while each node-candidate is chosen at most once (which is possible, since $k < n$), and some dummy voter wins in issue $k+1$. We will show that $\bar{w}^*$ leads to a strictly higher score than $\bar{w}$.

First, observe that, in both outcomes, since each time a node-candidate is selected exactly three edge-voters approve of it, the total satisfaction (ignoring the dummy voters w_i) will be $3k+1$ (the extra 1 comes from the last issue). Next, let $s = (s_1, \dots, s_{m+p})$ and $s^* = (s^*_1, \dots, s^*_{m+p})$ be the sorted satisfaction vectors (ignoring the dummy voters in $\{w_i\}_{i \in [2m-p]}$) when $\bar{w}$ and $\bar{w}^*$ are the outcomes, respectively. Furthermore, let i_1, i_2 and i_3 be the three smallest indexes such that $s_{i_1} = 1$, $s_{i_2} = 2$, and $s_{i_3} = 3$ hold. If any of these indexes is undefined, we set it to $m + p + 1$. Moreover, we define i^*_1 and i^*_2 analogously for s^* (observe that no voter here can have satisfaction greater than 2). Clearly $i^*_1 = p < i_1$, as $\bar{w}$ satisfies once at most $3m - p$ voters, whereas $\bar{w}^*$ satisfies once exactly

$3m - p + 1$ voters. We get that:

$$OWA_\alpha(\bar{w}) < OWA_\alpha(\bar{w}^*) \implies \alpha \cdot s < \alpha \cdot s^* \implies$$
$$\sum_{i=i_1}^{m+p} \alpha_i + \sum_{i=i_2}^{m+p} \alpha_i + \sum_{i=i_3}^{m+p} (s_i - 2)\alpha_i < \sum_{i=p}^{m+p} \alpha_i + \sum_{i=i_2^*}^{m+p} \alpha_i \implies$$
$$\sum_{i=i_2}^{m+p} \alpha_i + \sum_{i=i_3}^{m+p} \sum_{j=3}^{s_i} \alpha_i < \sum_{i=p}^{i_1-1} \alpha_i + \sum_{i=i_2^*}^{m+p} \alpha_i.$$

If $i_2^* < i_2$, we obtain:

$$\sum_{i=i_3}^{m+p} \sum_{j=3}^{s_i} \alpha_i < \sum_{i=p}^{i_1-1} \alpha_i + \sum_{i=i_2^*}^{i_2-1} \alpha_i.$$

Since $i_1 - 1 \leq i_2 - 1 < i_3$, every addend occurring on the left-hand side is smaller or equal to every addend occurring on the right side. In particular, α_p is positive and strictly greater than all the addends on the left side (as $p < i_3$). Furthermore, since $\sum_i s_i = \sum_i s_i^* = 3k + 1$, there is the same number of addends being summed on both sides. It follows that $OWA_\alpha(\bar{w}) < OWA_\alpha(\bar{w}^*)$. If, on the other hand, $i_2^* \geq i_2$, we obtain:

$$\sum_{i=i_2}^{i_2^*-1} \alpha_i + \sum_{i=i_3}^{m+p} \sum_{j=3}^{s_i} \alpha_i < \sum_{i=p}^{i_1-1} \alpha_i.$$

Since $i_1 - 1 < i_2$ and $i_1 - 1 < i_3$, by similar arguments as above, we conclude that $OWA_\alpha(\bar{w}) < OWA_\alpha(\bar{w}^*)$. But this is impossible, as we assumed that $\bar{w}$ is the outcome. We have finally reached the required contradiction: it cannot be that some edge-voter v_e loses all issues in $[k]$.

Now, let us show that c_{d_1} wins in $k + 1$ if all edge-voters win at least once in issue in $[k]$. If voter d_1 never won an issue in $[k]$, then it means she has a satisfaction of 0. Since all edge-voters and all the w_i won at least once, there are at least $m + 2m - p = 3m - p$ voters with a satisfaction of at least 1. Therefore, d_1 occupies a position within the first p entries of the satisfaction vector, whereas all edge-voters occupy a position within the last $4m - p$ entries. Since $\alpha_p > \alpha_{p+1}$, in this case choosing in issue $k + 1$ candidate c_{d_1} will yield a greater score than choosing a voter-candidate c_{v_e} for any edge $e \in E$. Finally, since c_{d_1} dominates in the tiebreaking every other candidate c_{d_j} in issue $k + 1$, here we must choose c_{d_1}. On the other hand, suppose that d_1 wins at least one issue $i \in [k]$. Suppose – towards a contradiction – that c_{d_1} is not selected in issue $k + 1$. Let c_v (for some voter $v \in N \setminus \{w_i\}_{i \in [2m-p]}$ distinct from d_1) be the candidate winning issue $k + 1$. Observe that if we make c_{d_1} win in issue $k + 1$ and make some candidate approved by v win in issue i, we would obtain a score that is higher or equal than before, and this would surely be preferred by tiebreaking: contradiction. We conclude that c_{d_1} must win in the final issue.

Finally, suppose that there exists no vertex cover for G with size at most k. Then, surely there is one edge-voter that never wins an issue in $[k]$ (otherwise, some vertex cover would exist). By tiebreaking, this edge-voter would decide the last issue, i.e., c_{d_1} would not win.

Second case. Here, we can assume that $\alpha_1 = \cdots = \alpha_p = 1 > \alpha_{p+1}$. We construct an instance $(\mathcal{E}, k + 1, c)$ of $\mathcal{R}$-Outcome-Determination. Here, we have one voter v_e for each edge $e \in E$, and three sets of dummy voters: $\{d_1, \ldots, d_{p-2m+1}\}$, $\{a_1, \ldots, a_m\}$, and $\{w_1, \ldots, w_{3m-p-1}\}$. In the first k issues, there is one candidate c_η for each node $\eta \in V$, plus one dummy candidate c_{d_i} for each dummy voter d_i. Here, each edge-voter v_e (with $e = \{\eta, \eta'\}$) approves of every node-candidate v_η where $\eta \notin e$. Furthermore, each dummy voter d_i approves only of dummy candidate c_{d_i}. Every other dummy candidate approves of all candidates. In the last issue, there are two candidates

c and c'. All dummy candidates d_i and w_i approve of both, every edge-voters approves only of c, while every dummy voter a_i approves only of c'.

We assume a tiebreaking mechanism almost identical to the one used in the proof of Theorem 16. However, here, in the last issue, c loses against c'.

First, suppose that there exists a vertex cover for G with size at most k. Since every dummy candidate c_{d_i} is always approved only by one voter and $\alpha_1 = \alpha_p = 1$, it is easy to see that any outcome where at least one such dummy candidate wins in the first k issues cannot have maximal score. Now, observe that, in total, the edge-voters will receive exactly a score of $k(m-3)$ for the first k issues (as every time we select some node-candidate, $m-3$ voters approve of it). This does not depend on which node-candidates we select; thus, to determine the outcome with the greatest score, we can focus on the last issue. First, observe that if the node-candidates selected in the first k issues correspond to a vertex cover, no edge-voter will have won more than $k-1$ issues within the first k issues (as every edge-voter loses at least once). Now, focusing on the last issue, note that the last $4m-p-1$ positions of the satisfaction vector will be occupied by dummy voters in $\{a_1, \dots, a_m\}$, and $\{w_1, \dots, w_{3m-p-1}\}$ (as they all have a satisfaction of at least k). Thus, if c wins in the last round, we get an extra score of $\sum_{i=p-2m+2}^{p-m+1} \alpha_i = m$; if c' wins, we get a score of $\sum_{i=p-m+2}^{p+1} \alpha_i = (m-1) + \alpha_{p+1}$. As $\alpha_{p+1} < 1$, here c wins. Similarly, if the node-candidates selected in the first k issues *do not* correspond to a vertex cover, we would still get a score of $(m-1) + \alpha_{p+1}$ for c' winning in the last round. So we have that c wins in the last issue if a vertex cover of size k exists.

Now, suppose that there exists no vertex cover for G with size at most k. Consider any outcome $\bar{w}$. Then, surely there is one edge-voter that wins all issues in $[k]$ (otherwise, if every edge-voter loses at least once, then some vertex cover would exist). In issue $k+1$, ignoring the voters supporting both candidates, both c and c' have exactly m voters supporting them, and in case either c or c' wins, the last $3m-p-1$ positions of the satisfaction vectors would be occupied by the dummy voters w_i (that have all satisfaction $k+1$). If c' wins, then we get an extra score of $\sum_{i=p-m+2}^{p+1} \alpha_i = (m-1) + \alpha_{p+1}$ (recall that all voters approving only of c' have a satisfaction of at least k, if we ignore the last issue). If c wins, we get at most the same score (as at least one voter has satisfaction k, she will occupy the $(p+1)$-position in the satisfaction vector). By tiebreaking, c cannot win in $\bar{w}$.

This concludes the proof. □

Theorem 18. *(Generalized-)$\mathcal{R}$-Free-Riding Recognition is* NP*-hard for every f-Thiele rule distinct from the utilitarian rule.*

Proof. We show the hardness of $\mathcal{R}$-Free-Riding-Recognition by a reduction from CubicVertexCover. Again, let ℓ be the smallest ℓ where $f(\ell) > f(\ell+1)$ holds, and consider an instance (G, k) of CubicVertexCover. Here, $k \in [m-1]$, and $G = (V, E)$ is an undirected graph with n nodes and m edges where every node has degree of 3. We construct an instance $(\mathcal{E}, \ell+k, c_{d_1}, d_2)$ of $\mathcal{R}$-Free-Riding-Recognition, where $\mathcal{E} = \langle N, \bar{A}, \bar{C} \rangle$ is an election with $(\ell+k)$ issues and $2m$ voters. We use a construction similar to the one in the proof of Theorem 16, except for the fact that, on issue $\ell+k$, voter d_2 approves only of c_{d_1}.

First, suppose that (G, k) is a yes-instance. By the same arguments used in the proof of Theorem 16, we know that c_{d_1} wins in issue $\ell+k$ (the fact that now d_2 also approves of it is irrelevant). Moreover, if d_2 votes for c_{d_2} in the last issue, we obtain the same election constructed in the proof of Theorem 16. We have already shown that here c_{d_1} wins the final issue: hence, d_2 can free-ride.

Now, suppose that (G, k) is a no-instance. Let us first show that c_{d_1} is still selected for issue $\ell+k$. Clearly, at least one edge-voter does not win any issue in $\ell, \dots, \ell+k$ (otherwise, a vertex cover would exist). Towards a contradiction, suppose that in $\mathcal{R}(\mathcal{E})$ some dummy candidate c_{d_j} is winning some issue $i \in \{\ell, \dots, \ell+k-1\}$. Then, at least one edge-voter must have satisfaction $\ell-1$ (for if all edge-voters were to have satisfaction at least ℓ, we could cover all but one edge with $k-1$ nodes). So let v_e be some edge-voter with satisfaction $\ell-1$ in $\mathcal{R}(\mathcal{E})$.

If we select c_η (for some $\eta \in e$) in i instead of c_{d_j}, we would increase the total score by at least $f(\ell)$ (contributed by v_e) and decrease it by $f(\ell^*)$ for some $\ell^* \geq \ell$ (contributed by d_j). Since $f(\ell^*) \leq f(\ell)$, this new outcome cannot have a lower score than $\mathcal{R}(\mathcal{E})$, and would be preferred to it by the tiebreaking. Contradiction: we conclude that no dummy voter d_j can win in $\ell, \dots, \ell + k - 1$. This implies, in particular, that neither d_1 nor d_2 win any issue in $\ell, \dots, \ell + k - 1$. Therefore, selecting c_{d_1} for issue $\ell + k$ contributes $2f(\ell)$ to the total score, whereas selecting any other candidate can contribute at most $f(\ell)$. Since $f(\ell) > f(\ell + 1) \geq 0$, c_{d_1} must win in the final issue.

It remains to show that d_2 cannot free-ride. Suppose that d_2 does not approve of c_{d_1}. Consider some edge-voter v_e that never wins in $\ell, \dots, \ell+k-1$ (which, as argued above, must exist). Choosing c_{v_e} in the final issue contributes at least $f(\ell)$ to the total score, whereas choosing c_{d_1} can contribute at most $f(\ell)$. By tiebreaking, c_{d_1} does not win in the final issue, that is, d_2 cannot free-ride.

To conclude, observe that the same construction can be used to show hardness for Generalized-$\mathcal{R}$-Free-Riding-Recognition. Indeed, here d_2 only approves of c_{d_1}, and hence free-riding and generalized free-riding coincide. □

Theorem 19. *(Generalized-)$\mathcal{R}$-Free-Riding Recognition is* NP*-hard for every α-OWA rule for which there is a $c \geq 3$ such that, for every $n \in \mathbb{N}$, there is a nonincreasing vector α of size ℓ (with $3n \leq \ell \leq cn$) such that $\alpha_1 > \alpha_\ell$ and $\alpha_{3n} > 0$.*

Proof. We show the hardness of $\mathcal{R}$-Free-Riding-Recognition by a reduction from CubicVertexCover. Consider an instance (G, k) of this problem. Here, $G = (V, E)$ is a graph with n nodes and m edges where each node has a degree of exactly three, and $k \in \mathbb{N}$. By the condition of the theorem, we know there is an $\ell \geq 3m$ (polynomial in the size of m) such that $\alpha = (\alpha_1, \dots, \alpha_\ell)$ contains at least $3m$ non-zero entries and $\alpha_1 > \alpha_\ell$. We will construct an instance of $\mathcal{R}$-Free-Riding-Recognition with $(k + 1)$ issues and ℓ voters. Since $\alpha_1 > \alpha_\ell$ and $\alpha_{3m} > 0$, there are two cases:

(1) There is a $p \in [2m]$ such that $\alpha_p > \alpha_{p+1}$ and $\alpha_{p+m} > 0$, or
(2) There is a $p \in \{2m + 1, \dots, \ell - 1\}$ such that $\alpha_1 = \alpha_p > \alpha_{p+1}$.

We treat them separately.

First case. We construct an instance $(\mathcal{E}, k + 1, v_{e^*}, c_{d_1})$ of $\mathcal{R}$-Free-Riding-Recognition (here, $e^* \in E$ is some edge, it does not matter which). The construction is similar to the one shown in the first case of the proof for Theorem 17. However, here, in issue $k + 1$ voter v_{e^*} approves only of c_{d_1}, and we have $\ell - m - p$ dummy voters w_i instead of $3m - p$. The latter change makes no difference in our construction.

First, note that $(\mathcal{E}, k + 1, v_{e^*}, c_{d_1})$ is indeed a legal instance of $\mathcal{R}$-Free-Riding-Recognition, as surely c_{d_1} wins in issue $k + 1$. If (G, k) is a yes-instance then we have already shown that this candidate wins, and here it is only receiving increased support. If it is a no-instance, then c_{d_1} will be supported by one voter that never won in the first k issues (namely, d_1), as well as by v_{e^*}. Since $\alpha_{p+m} > 0$ and since the edge-voters together with the dummy voters d_i occupy at most the first $p + m$ positions of the satisfaction vector, v_{e^*} will break the tie in favour of c_{d_1}.

Now, if (G, k) is a yes-instance of CubicVertexCover, then v_{e^*} can free-ride in the last issue: if she votes for her voter-candidate, then we have an election identical to the one constructed in the first case of the proof of Theorem 17, and we have already shown there that c_{d_1} wins if (G, k) has a vertex cover.

If (G, k) is a no-instance, then there are two cases: either v_{e^*} won in some issue in $[k]$ or not. If she did, there will be at least one voter v_e (with $e \in E \setminus \{e^*\}$) that never did, whose voter-candidate will get at least the same score as c_{d_1} (since v_{e^*} does not approve of the latter when she free-rides): c_{d_1} cannot win here. If she did not, there are again two cases: either v_{e^*} approves of some dummy candidate c_{d_i} (with $i > 1$) or of some c_{v_e} (where $e \in E$). In the first case, c_{d_i} would get a strictly higher score than c_{d_1}, while in the second case c_{v_e} would get at least the same score as c_{d_1} (and win by tiebreaking). In all cases, c_{d_1} loses: no free-riding is possible.

Second case. We construct another instance $(\mathcal{E}, k+1, a_1, c)$ of the problem $\mathcal{R}$-Free-Riding-Recognition. The construction is similar to the one shown in the second case of the proof for Theorem 17. However, here, in issue $k+1$ voter a_1 approves only of c, and we have $\ell - p - 1$ dummy voters w_i instead of $3m - p - 1$. The latter change makes no difference in our construction.

First, note that $(\mathcal{E}, k+1, a_1, c)$ is a legal instance of $\mathcal{R}$-Free-Riding-Recognition. Consider how the election is constructed, and recall that no dummy voter d_i can ever win here. In the last issue, c' receives the support of $m-1$ voters, whereas c receives the support of $m+1$ voters. Since the last $\ell - p - 1$ positions are occupied by voters w_i (who approve of all candidates and have a satisfaction of $k+1$), and since $\alpha_1 = \alpha_p = 1$, c wins here.

If a_1 free-rides in $k+1$, she must vote only for c'. Here, we obtain a construction identical to the one shown in the second case of the proof Theorem 17, and we have already shown there c wins if and only if (G, k) is a yes-instance.

Finally, observe that, in both cases, the same construction can be used to show hardness for Generalized-$\mathcal{R}$-Free-Riding-Recognition. Indeed, here the manipulator only approves of one alternative, and hence free-riding and generalized free-riding coincide. This concludes the proof. □

Theorem 20. *(Generalized-)$\mathcal{R}$-Free-Riding Recognition is* coNP*-hard for every α-OWA rule for which there is a $c \geq 2$ such that, for every $n \in \mathbb{N}$, there is a nonincreasing vector α of size ℓ (with $n < \ell \leq cn$) such that $\alpha_1 > \alpha_\ell$ and $\alpha_{\ell-n+1} = 0$.*

Proof. We show the hardness of $\mathcal{R}$-Free-Riding-Recognition by a reduction from VertexCover (Garey and Johnson 1979). Consider an instance of this problem, (G, k), where G has n nodes and m edges. By the condition of the theorem, we know there is an $\ell > m$ (polynomial in the size of m) such that $\alpha = (\alpha_1, \dots, \alpha_\ell)$ contains at least m zeros and at least one non-zero value. We will construct an instance $(\mathcal{E}, k+1, d_1, c_{d_1})$ of $\mathcal{R}$-Free-Riding-Recognition with ℓ voters and $k+1$ issues. Here, let p be the unique value such that $\alpha_p > \alpha_{p+1} = 0$.

In $\mathcal{E}$, there is one voter v_e for each edge $e \in E$, p dummy voters $d_1, \dots, d_p$, and $\ell - m - p$ dummy voters $w_1, \dots, w_{\ell-m-p}$. In all issues, all voters w_i approve of all candidates (so they always have satisfaction $k+1$, and occupy the last $\ell - m - p$ positions of the satisfaction vector). In the first k issues, there is one candidate c_η for each node $\eta \in V$, plus one candidate c_{d_i} for every dummy voter d_i. Here, each edge-voter v_e approves of the two node-candidates v_η and $v_{\eta'}$ such that $e = \{\eta, \eta'\}$, while every d_i approves of c_{d_i}. In the last issue, there is one candidate c, plus one candidate c_v for all voters $v \in N$, and any such v approves only of c_v. In the case of ties, we assume that in the last issue c_{d_1} dominates all candidates and that c_{d_1} is dominated by all other candidates in all other issues.

We will show that $(\mathcal{E}, k+1, d_1, c_{d_1})$ is a legal instance of $\mathcal{R}$-Free-Riding-Recognition (i.e., that c_{d_1} wins in issue $k+1$). Furthermore, we will show that (G, k) is a yes-instance if and only if $(\mathcal{E}, k+1, d_1, c_{d_1})$ is a no-instance.

Suppose (G, k) is a no-instance of VertexCover. We show that c_{d_1} wins in issue $k+1$ and that d_1 can free-ride here. First, observe that if any dummy candidate c_{d_i} wins in issue $k+1$, then the total satisfaction will be 0 (as we cannot give a satisfaction of at least 1 to all edge-voters in the other k issues, since (G, k) is a no-instance). On the other hand, if any candidate c_{v_e} (for some edge e wins), then the total satisfaction will still be 0: otherwise, that would mean that we could cover the remaining edges in $E \setminus \{e\}$ with $k - p$ nodes, but that is impossible (otherwise, we could cover all edges with k nodes). By tiebreaking, c_{d_1} wins in the final issue. Observe that if d_1 votes for c, we obtain the same effects: d_1 can free-ride.

Suppose (G, k) is a yes-instance of VertexCover. We show that c_{d_1} wins in issue $k+1$, but d_1 cannot free-ride here. Clearly, at least $m+1$ voters (ignoring the dummy voters w_i) need to win here: was not this the case, the total score would be zero, but the outcome where all edge-voters win in the first k issues and some dummy candidate wins in the last issue has a greater satisfaction (regardless of whether d_1 free-rides or not). If d_1 never wins in the first k issues, then it is clear that she must win in issue $k+1$: satisfying in this issue some voter that has never won surely will maximize the score (since this voter will be within the first p entries of the vector), and

c_{d_1} is preferred in the tiebreaking mechanism. If, on the other hand, c_{d_1} wins in some issue $i \in [k]$, but loses to some candidate c_v (with $v \neq d_1$) in issue $k+1$, then we could obtain an outcome with a score greater or equal by making c_{d_1} win in issue $k+1$, and making some candidate approved by v win in issue i. Since this would be preferred in the tiebreaking, c_{d_1} wins in $k+1$. Now, suppose that d_1 does not approve of c_{d_1} in issue $k+1$ (i.e., she attempts to free-ride). If c_{d_1} never wins in any issue in $[k]$, then clearly it cannot win in $k+1$: picking some candidate that now d_1 approves for in the last issue would give a greater score. If, on the other hand, c_{d_1} wins in some issue in $i \in [k]$ and also in $k+1$, we can obtain an outcome with a greater or equal score (and preferred in the tiebreaking) by making some node-candidate win in issue i and some candidate approved by d_1 in issue $k+1$. Therefore, d_1 cannot free-ride here.

Finally, observe that the same construction can be used to show hardness for Generalized-$\mathcal{R}$-Free-Riding-Recognition. Indeed, here d_1 only approves of c_{d_1}, and hence free-riding and generalized free-riding coincide. This concludes the proof. □

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