Beyond Lower Quota: Avoiding Overrepresentation in Multi-Winner Voting

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In the recent social choice literature on proportional representation, much attention has been given to the question of avoiding *underrepresentation* in approval-based multi-winner voting. In this paper, we explore the largely overlooked complementary question of avoiding *overrepresentation.* This has not been explored systematically, despite being a desirable property with concrete applications. Intuitively, overrepresentation happens when a group determines a disproportionately large part of the committee, thereby exceeding the group's *quota.*

We formulate a strong and appealing axiom for avoiding overrepresentation, called *justified upper quota* (JUQ). We introduce a generalization of Thiele rules, *composite Thiele rules,* and characterize the unique rule in this class satisfying our axiom. This rule, *Adams-AV*, which naturally extends Adams' apportionment method, has not been studied before. Additionally, we introduce a polynomial-time rule that satisfies JUQ.

Furthermore, we introduce *justified near quota*, an axiom that balances avoiding under- and overrepresentation. It characterizes the unique Thiele rule extending the Sainte-Laguë apportionment method. Finally, we analyze the compatibility of our axioms with established proportionality notions such as EJR+.

CCS Concepts: • **Computing methodologies** → *Multi-agent systems*; • **Applied computing** → *Economics*; • **Theory of computation** → **Algorithmic game theory and mechanism design**.

Additional Key Words and Phrases: Multi-Winner Voting, Approval Voting, Overrepresentation, Proportionality

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1 Introduction

Consider a scenario where a group of agents must jointly select a certain number of items from a pool of alternatives. Many real-world situations can be described in this way; for example, committee elections [Lackner and Skowron, 2023], parliamentary apportionment [Balinski and Young, 2001], or finding group recommendations [Skowron et al., 2016].

This problem, known as *multi-winner voting* [Faliszewski et al., 2017, Lackner and Skowron, 2023], has become one of the most-studied settings in (computational) social choice [Brandt et al.,

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2016]. Most of the effort has focused on *approval-based multi-winner voting*, where voters express their preferences by *approving* candidates [Lackner and Skowron, 2023]. In particular, one of the main issues in this area is how to guarantee *proportional representation*, which has so far largely been equated with avoiding underrepresentation. Briefly, the idea is that if a group of voters is "large enough", then it deserves to be represented in the outcome. More precisely, if a certain fraction of the voting population has similar preferences, they should be represented by (i.e., approve of) at least roughly the same fraction of candidates in the winning committee. Much progress has been made in this direction, and many properties (*axioms*) have been proposed that formally capture this notion (e.g., extended justified representation [Aziz et al., 2017], priceability [Peters and Skowron, 2020], and many others [Brill and Peters, 2023, Kalayci et al., 2025, Peters et al., 2021]). Furthermore, there are concrete voting rules that embody this property, for instance, Proportional Approval Voting (PAV) [Lackner and Skowron, 2021, Thiele, 1895], the Method of Equal Shares [Peters and Skowron, 2020], or Phragmén's sequential rule [Brill et al., 2024].

The main goal of our paper is to look at the other side of the coin, which has not received as much attention: *overrepresentation.* In short, this happens when a group of voters receives a fraction of the winning committee that is much larger than what its size would warrant. To illustrate this, consider the example shown in Figure 1. Here, we assume that voters and candidates correspond to two-dimensional points and voters prefer candidates close to them, a type of spatial model that is standard in voting theory. While the groups of voters are of equal size (50 each), there are 10 candidates associated with group 1 and 90 with group 2 (all points were sampled uniformly in either $[0.5, 1.5]^2$ or $[-1.5, -0.5]^2$). We are interested in selecting 10 candidates, and voters approve the 10 candidates closest to them. Consequently, the approval preferences of voters in group 1 are identical, whereas those in group 2 are diverse.[1] How do established, *proportional* voting rules handle this scenario? When averaging over 1000 sampled instances, group 2 receives on average 3.5 candidates with PAV, between 1.7 and 3.4 with the Method of Equal Shares (depending on the completion method), and 3.4 with Phragmén's sequential rule. We see that these proportional rules lead to *overrepresentation of group 1.*[2] In contrast, Sainte-Laguë Approval Voting, which plays a prominent role in our paper, treats both groups almost equally (4.9 for group 2).

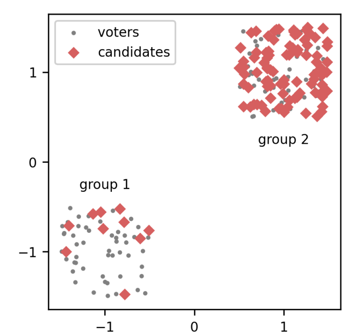

Fig. 1. Two groups with the same number of voters.

Depending on the application, avoiding overrepresentation can be as important as guaranteeing representation (or more). In parliamentary apportionment [Balinski and Young, 2001, Pukelsheim, 2014], a special case of multi-winner voting, avoiding overrepresentation is equivalent to what is known as *respecting upper quota* (or *upper ideal frame*). This notion has received much attention and can be seen as a way to favor small parties. Going in a more technical direction, Cevallos and Stewart [2021] highlight the connection between overrepresentation and security in distributed systems, particularly blockchain networks. More specifically, Boehmer et al. [2024] study overrepresentation in the context of *Polkadot*, a blockchain with a mechanism based on multi-winner voting, again

[1] A similar scenario can occur even when candidates are distributed evenly across the two groups, if one group—potentially for strategic reasons—coordinates to vote for the same 10 candidates. Everyone in this group submits the same ballot (a "bullet vote"), thus achieving high cohesiveness in the sense of justified representation, as defined by Aziz et al. [2017].

[2] One might wonder why this does not constitute an example of *under*representation for group 2. Indeed, the preferences of group 2 are too diverse to be regarded as a well-defined group with cohesive preferences; instead, there are many subgroups that are too small to be considered by established proportional rules as deserving of representation.

highlighting the link between overrepresentation and security.[3] Beyond multi-winner voting, Jerrett and Anshelevich [2025] recently studied the related problem of selecting a representative subset from a set of agents embedded in a metric space. They argue that solutions that overrepresent certain groups are undesirable and propose an axiom for avoiding them.

Despite the importance of this matter, previous work in approval-based multi-winner voting has mostly focused on avoiding underrepresentation. Indeed, rules that respect upper quota have received almost no attention. For example, it is known that PAV is the only rule in the class of *Thiele rules*, a normatively attractive family of rules, that satisfies strong guarantees against underrepresentation (such as extended justified representation [Aziz et al., 2017, Sánchez-Fernández et al., 2017]). No corresponding results are known for overrepresentation. Notably, in their survey book on approval-based multi-winner voting, Lackner and Skowron [2023] include the study of proportionality beyond underrepresentation as an open question. They call for further analysis of rules that avoid overrepresentation, as well as rules that generalize the Sainte-Laguë method of apportionment (which can be seen as a way to balance under- and overrepresentation) to the multi-winner setting. However, even defining a reasonable overrepresentation axiom for the multi-winner setting poses some conceptual difficulties (which we explore in Section 3).

Our contributions approach overrepresentation from several angles:

- We introduce a strong and appealing upper quota axiom, called *justified upper quota (JUQ)*. While this axiom is not satisfiable within the well-known class of Thiele rules, it can be satisfied in the class of *composite Thiele rules*. The latter is analogous to the class of *composite social choice scoring functions* in standard single-winner voting [Young, 1975]. Intuitively, composite Thiele rules are defined through a sequence of Thiele rules and work in stages. First, we compute the result of the first Thiele rule. Then, at each subsequent stage, we apply the next rule to break the ties of the previous stage. To the best of our knowledge, this is the first paper to introduce composite Thiele rules.
- We show that, among this general class, JUQ pinpoints a unique rule: *Adams-AV*. This is the natural extension to the multi-winner voting setting of an apportionment rule proposed by John Quincy Adams. The latter is notable as the only population monotone apportionment rule that respects upper quota [Balinski and Young, 2001], which, as discussed above, corresponds to avoiding overrepresentation in the apportionment setting. To the best of our knowledge, although Lackner and Skowron [2021] briefly mentioned this rule, we are the first to initiate the axiomatic study of Adams-AV. Finally, since computing Adams-AV corresponds to an NP-hard optimization problem, we also introduce a polynomial-time rule satisfying JUQ.
- Taking inspiration from the near quota axiom in the apportionment literature, we also introduce *justified near quota* (JNQ), a notion that captures both under- and overrepresentation by requiring each group's representation to deviate as little as possible from its quota. We show that Sainte-Laguë Approval Voting, a Thiele method based on the Sainte-Laguë method of apportionment (which satisfies the near quota axiom [Balinski and Young, 2001] in that setting), satisfies JNQ and is in fact the only composite Thiele rule with this property.
- Finally, we investigate the compatibility of upper, near, and lower quota axioms. We show that if we allow committees to be smaller than the target committee size k, we can guarantee a stronger version of JUQ, a stronger version of extended justified representation (EJR+, introduced by Brill and Peters [2023]) and JNQ simultaneously. We identify further conditions

[3] However, note that the rules considered by Cevallos and Stewart, maximin-support and Phragmén-maximin-support, later also considered by Boehmer et al., do not respect upper quota even in the apportionment setting. Indeed, maximin-support was originally introduced by Sánchez-Fernández et al. [2024] as a rule that generalizes D'Hondt's method of apportionment, which respects lower quota but not upper quota.

under which these axioms can be guaranteed jointly. Whether any two of JUQ, JNQ, and EJR+ are compatible for committees of size exactly k remains an intriguing open problem, and we show that some natural approaches to proving this compatibility cannot succeed.

This is an extended version of a paper presented earlier at a workshop [Lackner and Nardi, 2025]. All omitted proofs can be found in the full version of our paper [Baychkov et al., 2026].

2 Model and Voting Rules

In this section, we introduce the model of *approval-based multi-winner voting* [Lackner and Skowron, 2023] and its special case, the *apportionment* problem [Balinski and Young, 2001, Pukelsheim, 2014].

We denote by $\mathbb{N}$ the set of natural numbers including 0, i.e., the set $\{0, 1, 2, \ldots\}$. Given a positive integer $\ell \in \mathbb{N}_{>0}$, we define $[\ell] = \{1, \ldots, \ell\}$. Furthermore, given a set S, we denote by $\mathcal{P}(S)$ its powerset and for $\ell \in \mathbb{N}$ by $\mathcal{P}_\ell(S) = \{T \in \mathcal{P}(S) : |T| = \ell\}$ the set of subsets of size exactly ℓ.

2.1 Approval-Based Multi-Winner Voting

Let $N = [n]$ be the set of *voters* and let C be the set of *candidates* with $|C| = m > 0$. An *approval profile* (or simply *profile*) is a tuple $\boldsymbol{A} = (A_1, \ldots, A_n) \in \mathcal{P}(C)^n$, where $A_i \subseteq C$ for every $i \in N$. For a given voter $i \in N$, we call A_i the *approval ballot* of this voter. Given a candidate $c \in C$, we denote its set of *supporters* by $N(c) = \{i \in N : c \in A_i\}$ and its size by $n_c = |N(c)|$. For simplicity we make the (technical) assumption that $n_c > 0$ for all $c \in C$.

Further, we are given a *target committee size* $k \in [m]$. As every candidate is approved by at least one voter, we can uniquely identify an election by the approval profile $\boldsymbol{A}$ together with the committee size k (by taking $C = \bigcup_{i \in N} A_i$). We call $(\boldsymbol{A}, k)$ an *instance.* A *voting rule* is a function F that maps every instance $(\boldsymbol{A}, k)$ to a non-empty set of committees $\mathcal{W} \subseteq \{W \subseteq C : |W| \leq k\}$. Observe that we allow rules to return committees smaller than the target size; we say that F is *exhaustive* if it always returns committees of size k, i.e., if $F(\boldsymbol{A}, k) \subseteq \mathcal{P}_k(C)$ for all $(\boldsymbol{A}, k)$.[4] Given two voting rules F and G, we say that F is a *refinement* of G if $F(\boldsymbol{A}, k) \subseteq G(\boldsymbol{A}, k)$ for all inputs.

For a set of voters $N' \subseteq N$ and a target committee size k, we define its *quota* as $q(N') = k\frac{|N'|}{|N|}$. We also define $q_c = q(N(c))$ for a candidate $c \in C$. The quota of a group represents the (possibly fractional) number of seats on the committee that corresponds to the size of the group. Intuitively, N' would be entitled to fill this many seats on the committee if they can agree on sufficiently many candidates to do so. We call $\lceil q(N') \rceil$ the *upper quota* and $\lfloor q(N') \rfloor$ the *lower quota* of N'. In the next subsection, we present an example illustrating quotas (Table 1).

Ideally, we would like to guarantee that, for each group N', the committee contains at least $\lfloor q(N') \rfloor$ candidates approved by all voters in N'. As it is easy to see that this is impossible in general [see, e.g., Brill et al., 2025], several weaker lower quota axioms have been proposed in the literature [Aziz et al., 2017, Brill and Peters, 2023, Kalayci et al., 2025, Peters et al., 2021, Sánchez-Fernández et al., 2017]. We will focus on *extended justified representation plus* (EJR+), due to Brill and Peters [2023], a strong proportional representation axiom satisfiable in polynomial time. We express it in terms of quota—differently than how it was initially introduced—to match our notation.

Definition 1. Given a target committee size k, a set $W \subseteq C$ satisfies *EJR+* if there is no candidate $c \in C \setminus W$ and non-empty group of voters $N' \subseteq N(c)$ such that $|A_i \cap W| < \lfloor q(N') \rfloor$ for all $i \in N'$. △

That is, there should be no group N' of voters approving a common unchosen candidate such that everyone in this group approves fewer committee members than the group's lower quota.

[4]Note that exhaustiveness makes avoiding overrepresentation more difficult. Intuitively, the empty committee is not overrepresenting any group of voters (as no voter is represented at all).

2.2 Apportionment

The *apportionment* setting is a special case of the approval-based multi-winner voting model. Intuitively, in apportionment, candidates are partitioned into parties, and each voter approves all candidates of exactly one party; in this sense, voters are partitioned into parties as well. Formally, an instance $(\boldsymbol{A}, k)$ is an *apportionment instance*, if for every $i, j \in N$ we have either $A_i = A_j$ or $A_i \cap A_j = \emptyset$. Furthermore, for every $i \in N$ we must have $|A_i| = k$, i.e., we assume that each party has enough candidates to fill every seat. An *apportionment rule* is an exhaustive voting rule whose domain is restricted to apportionment instances. We say that a voting rule F *extends* an apportionment rule G if $F(\boldsymbol{A}, k) = G(\boldsymbol{A}, k)$ for all apportionment instances $(\boldsymbol{A}, k)$.

We introduce some additional notation for apportionment instances. We partition N into t disjoint equivalence classes called *parties*, that is, $N = P_1 \cup \cdots \cup P_t$. Two voters have the same ballot if and only if they belong to the same party. Given a committee W, we define the *seats* given to party P_ℓ as $a_\ell(W) = |A_i \cap W|$ (for some arbitrary $i \in P_\ell$). When W is fixed, we just write a_ℓ.

Definition 2. An apportionment rule satisfies *apportionment upper quota* (resp., *lower quota*) if, for every outcome W selected by this rule and party P_ℓ, we have that $a_\ell \leq \lceil q(P_\ell) \rceil$ (resp., $a_\ell \geq \lfloor q(P_\ell) \rfloor$). Moreover, an apportionment rule satisfies *apportionment near quota* if for every outcome W selected by this rule there do not exist two parties P_i and P_j such that $|q(P_i) - (a_i - 1)| < |q(P_i) - a_i|$ and $|q(P_j) - (a_j + 1)| < |q(P_j) - a_j|$. In other words, we should not be able to take a seat from party P_i and allocate it to P_j while bringing both parties closer to their respective quotas. △

We refer the reader to the book of Balinski and Young [2001] for an extended discussion of lower, upper, and near quota in the apportionment setting.

Example 1. To illustrate the three different quota concepts, consider the apportionment instance depicted in Table 1. In the instance we abstract from the underlying voters and candidates and consider each party only by its vote count, and each outcome only by its seat allocation. For the instance, we assume $k = 11$ and have $n = 110$. First, allocation $a^{(1)}$ satisfies lower and upper quota. It does not, however, satisfy near quota, as reallocating one seat from party P_1 to party P_2 would decrease their respective distances to their quotas by 0.2 each. Allocation $a^{(2)}$ satisfies upper quota and near quota. It does not, however, satisfy lower quota due to party P_1. Allocation $a^{(3)}$ satisfies lower quota and near quota, but not upper quota, due to party P_2. Finally, allocation $a^{(4)}$ satisfies all three. △

	P_1	P_2	P_3	P_4
$\lvert P_i \rvert$	24	36	35	15
$q(P_i) = k\lvert P_i\rvert/n$	2.4	3.6	3.5	1.5
$\lfloor q(P_i) \rfloor$	2	3	3	1
$\lceil q(P_i) \rceil$	3	4	4	2
$a_i^{(1)}$	3	3	3	2
$a_i^{(2)}$	1	4	4	2
$a_i^{(3)}$	2	5	3	1
$a_i^{(4)}$	2	4	4	1

Table 1. Apportionment instance depicting differences in quota concepts ($n = 110$ and $k = 11$).

It is easy to see that for apportionment instances, EJR+ and lower quota coincide.

Observation 1. *For apportionment instances, EJR+ is equivalent to apportionment lower quota.*

As a final ingredient of the apportionment setting, we introduce *divisor methods*, a normatively appealing class of apportionment rules [Balinski and Young, 2001].[5]

[5]Divisor methods are the only apportionment rules that are *population monotone*. Population monotonicity captures the idea that if a party P_i's size increases while P_j's decreases, then a_i should not decrease while a_j increases. Balinski and Young [2001] define divisor methods in a slightly different but equivalent way (see Proposition 3.3 of their book).

Definition 3. A divisor method is defined by a function $d\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ with $d(i) \leq d(i+1)$ for every $i \in \mathbb{N}$. We sometimes write d as a vector. This method proceeds sequentially, starting with $W = \emptyset$ and iteratively allocating the seat to a party P_ℓ maximizing $|P_\ell|/d(a_\ell)$ by adding an arbitrary unchosen candidate approved by this party to W. This is repeated until $|W| = k$, and the method returns all committees that can be obtained by the above procedure plus *some* way of breaking ties. We adopt the convention that $i/0 = \infty$ for all $i \in \mathbb{N}_{>0}$ and that $i/0 > j/0$ if $i > j$. △

We highlight three well-known divisor methods which are relevant to our analysis:

- *D'Hondt's* (or *Jefferson's*) *method* is given by $d_H = (1,\ 2,\ 3,\ \ldots)$;
- *Adams' method* is given by $d_A = (0,\ 1,\ 2,\ \ldots)$;
- *Sainte-Laguë's* (or *Webster's*) *method* is given by $d_S = (1,\ 3,\ 5,\ \ldots)$.

These three methods are notable as they are the only divisor methods that satisfy, respectively, lower, upper, and near quota in apportionment [Balinski and Young, 2001]. Note that, since $d_A(0) = 0$ and $s/0 = \infty$ for all $s \in \mathbb{N}_{>0}$, Adams' method starts by assigning one seat to each party ordered by their size (until all parties are assigned at least one seat or the committee size k is reached) and then proceeds similarly to D'Hondt's method.

2.3 Multi-Winner Voting Rules

Since Adams' and Sainte-Laguë's methods satisfy upper quota and near quota, respectively, in apportionment, generalizing them to multi-winner voting is a natural first step towards understanding overrepresentation in this setting. Most divisor methods have direct counterparts in the popular class of *Thiele rules*, introduced by Thorvald N. Thiele [Janson, 2016, Thiele, 1895]. These (exhaustive) rules are well-studied in multi-winner voting [Lackner and Skowron, 2023]. Intuitively, each Thiele rule is defined by a *weight function* which determines the score that a voter assigns to a committee given the number of candidates they approve in that committee.

However, Adams' method does not correspond directly to a Thiele rule, as we will discuss later. Thus, we introduce a more general class of rules, called *composite Thiele rules.* A composite Thiele rule is defined by a sequence of Thiele rules. First, we select a set of winning committees via the first rule. Then, we select a subset of these committees through the second rule. This process continues until either every rule in the sequence has been applied, or we reach a fixed point.

Definition 4. A *weight function* is a function $w\colon \mathbb{N}_{>0} \to \mathbb{R}_{\geq 0}$ such that for every $s \in \mathbb{N}_{>0}$, $w(s) \geq w(s+1)$. We often write weight functions as vectors and normalize $w(1) = 1$. Given $\boldsymbol{A}$, k, and $i \in N$, we define $\text{score}_w(A_i, W) = \sum_{s=1}^{|A_i \cap W|} w(s)$ and $\text{score}_w(\boldsymbol{A}, W) = \sum_{j \in N} \text{score}_w(A_j, W)$. The *simple* or *1-composite Thiele rule* corresponding to w is then $F_w(\boldsymbol{A}, k) = \arg\max_{W \in \mathcal{P}_k(C)} \text{score}_w(\boldsymbol{A}, W)$. Given two weight functions w_1 and w_2, we define the composition $\left(F_{w_2} \circ F_{w_1}\right)(\boldsymbol{A}, k) = \arg\max_{W \in F_{w_1}(\boldsymbol{A},k)} \text{score}_{w_2}(\boldsymbol{A}, W)$. Then, finally, for any sequence $\boldsymbol{w} = (w_1, w_2, \ldots)$ of length $\ell \in \mathbb{N}_{>0} \cup \{\infty\}$ of weight functions, the corresponding *ℓ-composite Thiele rule* is $F_{\boldsymbol{w}}(\boldsymbol{A}, k) = \bigcap_{i=1}^{\ell} (F_{w_i} \circ \cdots \circ F_{w_1})(\boldsymbol{A}, k)$. Note that if $\boldsymbol{w} = (w_1, \ldots, w_\ell)$ is finite the above collapses to $F_{\boldsymbol{w}}(\boldsymbol{A}, k) = \left(F_{w_\ell} \circ \cdots \circ F_{w_1}\right)(\boldsymbol{A}, k)$ and that this is well-defined even for $\ell = \infty$ as there are only finitely many committees. △

We allow for ∞-composite Thiele rules to also capture rules such as the one given by the infinite sequence $(1,\ 0,\ 0,\ \ldots)$, $(1,\ 1,\ 0,\ 0,\ \ldots)$, $(1,\ 1,\ 1,\ 0,\ 0,\ \ldots)$, etc. This is an analogue to the leximin social welfare function [Rawls, 1971, Sen, 2017].[6]

To illustrate our definition further, we present some standard Thiele voting rules:

[6] Observe that, since we assume n to be fixed, we do not technically need infinite sequences to capture this rule. However, we stick with this definition (which accommodates for electorates of variable size) as our results also hold for this more general case.

- *Approval Voting* (AV) is the Thiele rule given by $w = (1, 1, 1, \ldots)$.
- *Chamberlin-Courant* (CC) is the Thiele rule given by $w = (1, 0, 0, \ldots)$.
- *Proportional AV* (PAV) is the Thiele rule given by $w = (1, 1/2, 1/3, \ldots)$.

Let us give an intuitive idea of these rules. AV picks the committees with the highest total approval score. On the other hand, CC selects the committees that cover as many voters as possible, where a voter is covered if they approve at least one candidate in the committee (the score of a committee does not increase if a voter approves more than one candidate). One can think of AV and CC as lying on two different ends of the spectrum of (simple) Thiele rules: while AV is entirely utilitarian, CC tries to satisfy all voters whenever possible. PAV lies, in some sense, in between these two. Given a committee W, under PAV each voter assigns to W a score corresponding to the $|A_i \cap W|$-th harmonic number; the committees with the highest total score are selected. Thus any additional candidate a voter approves in the committee will yield diminishing returns.

There is a close connection between divisor methods and Thiele rules: the divisor method given by d is extended uniquely among simple Thiele rules by the one defined by $w(s) = d(0)/d(s-1)$ [Brill et al., 2018, Lackner and Skowron, 2021]. Note that this is not well-defined for divisor methods with $d(0) = 0$, such as Adams' method. We note that PAV uniquely extends D'Hondt's method and is the only Thiele rule that satisfies EJR+ [Aziz et al., 2017, Brill and Peters, 2023]. Since we are interested in upper and near quota notions, it is then natural to look at the Thiele rules that extend Adams' and Sainte-Laguë's methods. For Sainte-Laguë, the corresponding Thiele rule is:

Definition 5. Sainte-Laguë-AV (SLAV) is the simple Thiele rule given by $w = (1, 1/3, 1/5, \ldots)$. △

In contrast, Adams' method does not have a corresponding Thiele method, as its divisor sequence starts with 0 and hence would imply $w(1) = \infty$. This behavior, however, can be captured in the class of composite Thiele rules.

Definition 6. Adams-AV is the 2-composite Thiele rule given by $w_1 = (1, 0, 0, \ldots)$ and $w_2 = (1, 1, 1/2, 1/3, \ldots)$. △

Observe that Adams-AV first maximizes the number of voters with at least one candidate in the committee (just as CC does), then proceeds with a PAV-like weight function. This resembles exactly the behavior of Adams' method and indeed, we can verify that Adams-AV extends Adams' method.

Observation 2. *Adams-AV extends Adams' method.*

Returning to the example from the introduction (Figure 1), Adams-AV strongly advantages the diverse group (group 2) with 7.5 chosen candidates on average.[7] This further highlights the connection to CC, which is known to prioritize *diversity* (i.e., it is representation-focused) [Faliszewski et al., 2017, Lackner and Skowron, 2020]. To the best of our knowledge, Adams-AV has not been studied previously in the literature.

Example 2. Consider profile $\boldsymbol{A} = (\{a, b, c, d\}, \{a, b, c, d\}, \{a, b, c, e\}, \{d\}, \{b, c, e\}, \{c, e\})$ and let $k = 2$. Here, AV picks only $\{b, c\}$, the two most approved candidates. CC selects $\{c, d\}$ and $\{d, e\}$, as they both cover all voters. PAV selects $\{b, c\}$ and $\{c, d\}$. Among the two CC-optimal committees, Adams-AV picks only committee $\{c, d\}$. Similarly, SLAV picks only committee $\{c, d\}$. △

[7] One may wonder whether this is actually an *overrepresentation* of group 2, and indeed, this is a fair criticism. Adams-AV considers the small diverse groups within group 2 as distinct in the same way as PAV considers them distinct to justify ignoring them. Consequently, following the logic of this example, group 1 and 2 are only treated equally if both under- and overrepresentation are avoided.

3 Avoiding Overrepresentation

In the previous section, we introduced Adams-AV, a multi-winner rule that extends Adams' apportionment method. However, it is not immediately clear whether this rule always avoids overrepresentation, or even how we should extend the idea of overrepresentation to multi-winner elections. In this section, we answer both of these questions by generalizing the normative notion of an upper quota of parties to groups of voters in the approval-based multi-winner setting.

3.1 Justified Upper Quota

Let us begin by defining a natural upper quota analog of EJR+ by requiring that there is no selected candidate whose supporters all receive more than this candidate's upper quota. Note that, in contrast to EJR+, we only consider the full set of supporters $N(c)$ and not its subsets, as there could exist small subsets that are overrepresented while the entire set of approvers is underrepresented.

Definition 7. A set $W \subseteq C$ of candidates satisfies *upper quota* (UQ) if there is no selected candidate $c \in W$ with $|A_i \cap W| > \lceil q_c \rceil$ for all $i \in N(c)$. △

Note that to be overrepresented *everyone* needs to be over the quota, similarly to how EJR-style axioms require all members of a particular voter set to be below quota for underrepresentation to occur. UQ is trivially satisfiable by non-exhaustive committees, in particular by the empty committee.[8] However, it can be infeasible for exhaustive committees.

Example 3. Consider the profile $\boldsymbol{A} = (\{a, b, c\}, \{d, e, f\}, \{g\})$ and let $k = 6$. This is *almost* an apportionment instance, but here voters approve fewer than k candidates. Any exhaustive committee must contain either three candidates approved by voter 1 or three candidates approved by voter 2 (or both). Suppose, without loss of generality, that $\{a, b, c\} \subseteq W$. Then we have that $|A_1 \cap W| = 3 > \lceil q_c \rceil = 2$, violating UQ. A similar result holds for any other exhaustive committee. △

Due to the closeness of Example 3 to the apportionment setting, it seems hard to argue with the interpretation that voter 1 and 2 form a unique party or group each and that, if we select three candidates they approve, they are overrepresented. Thus, we must conclude that in some cases overrepresentation is unavoidable for exhaustive committees. However, we only want to permit overrepresentation if it is—in some sense—necessary. We interpret this necessity as there being no other group that could receive a candidate without any of the voters breaching their upper quota. We call such violations of upper quota "justified" and introduce the axiom *justified upper quota.*

Definition 8. Committee W satisfies *justified upper quota* (JUQ) if, whenever there is a $c \in W$ such that

$$|A_i \cap W| > \lceil q_c \rceil \quad \text{for all } i \in N(c),$$

then there exists no $d \in C \setminus W$ and nonempty $N' \subseteq N(d)$ such that

$$|A_i \cap ((W \cup \{d\}) \setminus \{c\})| \leq \lceil q(N') \rceil \quad \text{for all } i \in N'. \qquad \triangle$$

Note that we allow for subsets $N' \subseteq N(d)$ in our definition: for us it is sufficient that there exists some non-empty subgroup of the approvers of d who could be awarded an extra candidate without breaching their upper quota. Observe that UQ implies JUQ, and that we require $N' \neq \emptyset$ as otherwise the second condition would be vacuously true for any unselected candidate.

Even when every exhaustive committee overrepresents some group according to UQ, JUQ can remain meaningfully satisfiable.

[8] Note that q_c is defined in terms of the target committee size k, not the actual size of W.

Example 4. Let us revisit the scenario of Example 3, where $\boldsymbol{A} = (\{a, b, c\}, \{d, e, f\}, \{g\})$ and $k = 6$. First, consider the committee $W_1 = \{a, b, c, d, e, g\}$. This committee satisfies JUQ. Indeed, consider, for example, $c \in W_1$. We have that $\lceil q_c \rceil = 2$, but voter 1 has satisfaction 3, that is, she approves 3 candidates in the committee. However, the only candidate not in W_1 is f, and we have that, if we were to replace c with f in W_1, then voter 2 would get a satisfaction of $3 > 2 = \lceil q_f \rceil$. On the other hand, the committee $W_2 = \{a, b, c, d, e, f\}$ violates JUQ. Indeed, again c is a witness for an upper quota violation. However, now we can swap c with g, as $\lceil q_g \rceil = 2 \geq |A_3 \cap ((W_2 \cup \{g\}) \setminus \{c\})|$. △

Note that JUQ only disallows overrepresentation in the most egregious cases: if *all* members in a group $N(c)$ are above their upper quota, and there is another group $N' \subseteq N(d)$ where *all* members would be at most at their upper quota after replacing c with d, then JUQ is violated. Note the asymmetry: it would perhaps be more natural to require only *some* members of N' to be at most their upper quota to warrant the replacement of c by d. However, there are reasons to believe that this *symmetric variant* of JUQ is too strong, as we will discuss at the end of Section 3.2. We will therefore focus on JUQ and leave a more in-depth analysis of this stronger notion to future work.

Next, similarly to the relationship between EJR+ and apportionment lower quota, we can show that JUQ and apportionment upper quota are equivalent on apportionment instances.

Proposition 3. *For apportionment instances, JUQ is equivalent to apportionment upper quota.*

Proof. Consider an apportionment instance $(\boldsymbol{A}, k)$ and consider any committee W of size at most k. First, assume that W violates JUQ. Then there exists a candidate $c \in W$ (belonging to a party P_ℓ) such that $\lceil q(P_\ell) \rceil = \lceil q_c \rceil < |A_i \cap W| = a_\ell$ and hence W also violates apportionment upper quota. For the other direction, we observe that by definition of the quota it holds that $\sum_{\ell=1}^{t} a_\ell \leq k = \sum_{\ell=1}^{t} k \cdot \frac{|P_\ell|}{n} = \sum_{\ell=1}^{t} q(P_\ell)$. Hence, if there exists a party P_ℓ with $a_\ell > q(P_\ell)$ there must exist a second party P_s with $a_s < q(P_s)$ and thus as a_s is an integer $a_s + 1 \leq \lceil q(P_s) \rceil$. Hence, if apportionment upper quota is violated, we can take any selected candidate from party P_ℓ as c, any unselected candidate from party P_s as d and P_s as N' as a witness that JUQ is also violated. □

We remark that JUQ is verifiable in polynomial time (as is the case for EJR+ [Brill and Peters, 2023], but not the original EJR notion [Aziz et al., 2017]).

Proposition 4. *Whether a committee satisfies JUQ is decidable in polynomial time.*

3.2 Characterization of Adams-AV

In the previous section, we introduced JUQ as our upper quota axiom. It is easy to see that we can always satisfy JUQ via a non-exhaustive committee: for instance, the empty committee trivially satisfies it. However, similar to UQ, it is not obvious that there always exists a committee of size k satisfying JUQ. We show that this is always the case. In particular, we prove that every committee selected by Adams-AV satisfies JUQ.

Theorem 5. *Any refinement of Adams-AV satisfies justified upper quota.*

Proof. Let W be a committee of size k not satisfying JUQ. Our goal is to show that W cannot be selected by Adams-AV. Let $w_1 = (1, 0, 0, \ldots)$ and $w_2 = (1, 1, 1/2, 1/3, \ldots)$ be the two weight vectors used to define Adams-AV. Assume that there is a JUQ violation witnessed by the two candidates $c \in W$ and $d \in C \setminus W$, as well as by $N' \subseteq N(d)$. We consider the committee $W' = (W \setminus \{c\}) \cup \{d\}$, swapping the two candidates. Hence, our goal becomes to show that either

(1) $\text{score}_{w_1}(\boldsymbol{A}, W') > \text{score}_{w_1}(\boldsymbol{A}, W)$ or
(2) $\text{score}_{w_2}(\boldsymbol{A}, W') > \text{score}_{w_2}(\boldsymbol{A}, W)$ and $\text{score}_{w_1}(\boldsymbol{A}, W') = \text{score}_{w_1}(\boldsymbol{A}, W)$.

Hence, performing the swap increases the "score" associated with Adams-AV and the committee W could not have been selected by it.

Since c, d, and N' witness the JUQ violation it holds that $|A_i \cap W| > \lceil q_c \rceil$ for all $i \in N(c)$ and $|A_i \cap W'| \leq \lceil q(N') \rceil$ for all $i \in N'$. First, we notice that since $|A_i \cap W| > \lceil q_c \rceil$ it must particularly hold that $|A_i \cap W| > 1$ for all $i \in N(c)$. After removing c from W every voter in $N(c)$ still approves at least one candidate of W'. Thus, if there exists a voter $i \in N(d)$ with $A_i \cap W = \emptyset$ the score$_{w_1}$ would strictly increase as we cover more voters. If, however, $A_i \cap W \neq \emptyset$ for all $i \in N(d)$, we would get $\text{score}_{w_1}(\boldsymbol{A}, W') = \text{score}_{w_1}(\boldsymbol{A}, W)$.

Thus, for the second part we can assume that $A_i \cap W \neq \emptyset$ for all $i \in N(d)$ (and thus also for all $i \in N'$). Now, our goal is to lower bound the score difference $\text{score}_{w_2}(\boldsymbol{A}, W') - \text{score}_{w_2}(\boldsymbol{A}, W)$ and to show that this difference is always positive. First, we notice that for any $i \in N(c) \cap N(d)$ or $i \in N \setminus (N(c) \cup N(d))$ the number of approved candidates is the same in W' and W. Hence, for such a voter $\text{score}_{w_2}(A_i, W') - \text{score}_{w_2}(A_i, W) = 0$. We note that for the w_2 score vector, for a voter with $|A_i \cap W| \geq 1$ approved candidates on the committee, the marginal increase of adding another approved candidate is $\frac{1}{|A_i \cap W|}$, while the marginal loss of removing one is $\frac{1}{|A_i \cap W|-1}$. For any $i \in N(d) \setminus N(c)$ we know by our assumption that i approves at least one candidate in W. Hence, the score increases by $\text{score}_{w_2}(A_i, W') - \text{score}_{w_2}(A_i, W) = \frac{1}{|A_i \cap W|}$. Similarly, for any $i \in N(c) \setminus N(d)$ the score must decrease by $\text{score}_{w_2}(A_i, W') - \text{score}_{w_2}(A_i, W) = -\frac{1}{|A_i \cap W|-1}$. We get that

$$\begin{aligned}
\text{score}_{w_2}(\boldsymbol{A}, W') - \text{score}_{w_2}(\boldsymbol{A}, W) &= \sum_{i \in N(d) \setminus N(c)} \frac{1}{|A_i \cap W|} - \sum_{i \in N(c) \setminus N(d)} \frac{1}{|A_i \cap W| - 1} \\
&\geq \sum_{i \in N' \setminus N(c)} \frac{1}{|A_i \cap W|} - \sum_{i \in N(c) \setminus N'} \frac{1}{|A_i \cap W| - 1}.
\end{aligned}$$

For any $i \in N' \setminus N(c)$, since N' and d witness the JUQ violation, we know that $|A_i \cap W| + 1 \leq \lceil q(N') \rceil$ and therefore $|A_i \cap W| < q(N')$. Similarly, for any $i \in N(c) \setminus N'$ it must hold that $|A_i \cap W| \geq q_c + 1$. Further, we note that $N' \setminus N(c)$ must always be non-empty as otherwise $N' \subseteq N(c)$ and thus $q(N') \leq q_c$. However, then for any $i \in N'$ we would get $\lceil q(N') \rceil \leq \lceil q_c \rceil < |A_i \cap W|$, a contradiction to the second condition of JUQ. Thus, we obtain

$$\begin{aligned}
&\sum_{i \in N' \setminus N(c)} \frac{1}{|A_i \cap W|} - \sum_{i \in N(c) \setminus N'} \frac{1}{|A_i \cap W| - 1} > \sum_{i \in N' \setminus N(c)} \frac{1}{q(N')} - \sum_{i \in N(c) \setminus N'} \frac{1}{q_c} \\
&\qquad = \frac{|N' \setminus N(c)|}{q(N')} - \frac{|N(c) \setminus N'|}{q_c} = \frac{|N'|}{q(N')} - \frac{|N(c)|}{q_c} - \frac{|N' \cap N(c)|}{q(N')} + \frac{|N' \cap N(c)|}{q_c}.
\end{aligned}$$

Now, by definition of the quota $\frac{|N'|}{q(N')} = \frac{n}{k}$ and $\frac{|N(c)|}{q_c} = \frac{n}{k}$. Hence, this expression simplifies to

$$\frac{|N'|}{q(N')} - \frac{|N(c)|}{q_c} - \frac{|N' \cap N(c)|}{q(N')} + \frac{|N' \cap N(c)|}{q_c} = |N' \cap N(c)| \left(\frac{1}{q_c} - \frac{1}{q(N')} \right).$$

If $N' \cap N(c) = \emptyset$ this is 0 and hence $\text{score}_{w_2}(\boldsymbol{A}, W') - \text{score}_{w_2}(\boldsymbol{A}, W) > 0$ would hold. Otherwise, there must exist a voter $j \in N' \cap N(c)$. For this voter, we know that $\lceil q_c \rceil < |A_j \cap W| = |A_j \cap W'| \leq \lceil q(N') \rceil$. Thus, $q_c < q(N')$ and $(\frac{1}{q_c} - \frac{1}{q(N')})$ must be positive. Consequently, $\text{score}_{w_2}(\boldsymbol{A}, W') - \text{score}_{w_2}(\boldsymbol{A}, W) > 0$ also holds and W could not have been selected by Adams-AV. □

We complement this by a characterization of Adams-AV among composite Thiele rules: A composite Thiele rule satisfies apportionment upper quota if and only if it is a refinement of Adams-AV.

Theorem 6. *A composite Thiele rule satisfies apportionment upper quota if and only if it is a refinement of Adams-AV.*

Note that Theorem 6 can *in principle* be derived from the theory of apportionment, at least when it comes to a class of simple Thiele rules with strictly positive weights. Indeed, we know that every such simple Thiele rule extends a unique divisor method [Brill et al., 2018, Lackner and Skowron, 2021], and that all divisor methods except Adams' violate apportionment upper quota [Balinski and Young, 2001] (at least in the definition of Balinski and Young [2001] which forces $d(i) \in [i, i+1]$—with appropriate normalization—for any divisor method with function d). However, for non-simple Thiele rules and those not fitting the definition of Balinski and Young, this does not follow. We thus give a direct proof of the result (in the full paper [Baychkov et al., 2026]).

We further observe that Adams-AV fails the stronger symmetric variant of JUQ discussed in the previous section. Consider an 8-voter profile $\boldsymbol{A} = (\{a\}, \{a\}, \{a, d\}, \{a, b, c\}, \{a, b, c\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, d\})$ and let $k = 3$. Adams-AV here picks only $\{a, b, c\}$. However, $N(c)$ is overrepresented as all its members have satisfaction $3 > 2 = \lceil q_c \rceil$. If we switch c with d, then not all voters in $N(d)$ would be overrepresented as the voter with ballot $\{a, d\}$ would have satisfaction $2 = \lceil q_d \rceil$. Hence, Adams-AV and any refinement of it violate the symmetric variant of JUQ. By Theorem 6, the same holds for every other composite Thiele rule, motivating our focus on JUQ. However, observe that the rule we introduce later on (Section 3.4), UQER, would satisfy the stronger symmetric variant of JUQ as a close inspection of the proof of Lemma 9 shows.

In light of Theorem 6 one might wonder whether Thiele rules, which maximize satisfaction, are just ill-suited to avoiding overrepresentation or whether JUQ is just hard to satisfy. The latter seems to be the case, as, to the best of our knowledge, no other known multi-winner voting rule satisfies JUQ. Indeed, in the full version of the paper [Baychkov et al., 2026], we show counterexamples for a wide set of familiar rules (including Phragmén's rules and the method of equal shares, as well as all rules in the `abcvoting` Python package [Lackner et al., 2023] at the time of writing). Interestingly, even the sequential formulation of Adams-AV fails JUQ.

This is particularly important, as computing a winning committee under Adams-AV is NP-hard because it refines the CC rule, which is NP-hard to compute [Procaccia et al., 2008].[9] Thus, it is not clear whether we can find an exhaustive committee satisfying JUQ in polynomial time. In the next section, we try to tackle this question by investigating the dynamics of exchanging pairs of candidates that witness a violation of JUQ.

3.3 Swap Dynamics of JUQ

We can think of JUQ as being defined in terms of *swaps* of candidates, i.e., if there is a "bad" candidate c, we should swap it with the "good" candidate d (whenever possible). Formally, a JUQ swap for a committee W is a distinct pair of candidates $(c, d) \in C^2$ such that $c \in W$ and $d \notin W$ (together with some $N' \subseteq N(d)$) are witnesses to the violation of JUQ of W.

It is then natural to study the dynamics of such swaps to analyze the complexity of constructing an exhaustive committee that satisfies JUQ. Indeed, if we show a polynomial bound on the length of all possible sequences of JUQ swaps, then we have effectively shown the polynomial-time computability of JUQ. At the same time, studying the dynamics of JUQ swaps might give us further insights into the behavior of our axiom.

One first approach would be to show that, for any starting committee W, there is no sequence of JUQ swaps of form $(c_1, d_1), (c_2, d_2), \ldots, (c_{\ell-1}, d_{\ell-1}), (c_\ell, c_1)$, i.e., that it is never possible to add a candidate c back in the committee after removing it through a JUQ swap. This would immediately

[9]Indeed, hardness of CC is formulated by Procaccia et al. [2008] as: "Is there a committee W with a CC-score of at least s?". Since Adams-AV refines CC, a polynomial-time algorithm for Adams-AV would solve this question.

give us a polynomial-time algorithm for computing an exhaustive JUQ committee: we could start from an arbitrary exhaustive committee W, perform at most $(|C| - k)$-many swaps, and find a committee that satisfies JUQ. Unfortunately, this approach does not work in general.

Example 5. Consider the profile $\boldsymbol{A} = (\{a, b, c\}, \{a, b, d\}, \{e, f\}, \{e, f\})$ and let $k = n$. If we start with the committee $\{a, b, c, d\}$ and swap first a with e and then c with f, then (d, a) will be a JUQ swap, and hence we can add a back in. △

In the full version of the paper [Baychkov et al., 2026], we show that the above problem persists even assuming some clever schemata on how to pick which JUQ swap to perform next.

Another approach would be to use a bounded *potential function*, and show that each swap decreases this function. As there are only finitely many committees, this would show that there are no infinite sequences of JUQ swaps. To obtain a polynomial-time algorithm one would then need to show that such sequences of swaps are polynomially-bounded. Given the scope of the paper, a natural starting point for such a potential would be to consider some measure of "overrepresentation" of a committee. To name a few natural candidates, one might consider the number of candidates c for which $N(c)$ is overrepresented, the number of unique voters that are overrepresented, or the magnitude of total overrepresentation (i.e., sum of the satisfactions above quota).

Unfortunately, this approach also does not work—there are JUQ swaps that *increase* overrepresentation. Consider the profile $\boldsymbol{A} = (\{a, b, c\} \times 4,\ \{d, e, f\} \times 3,\ \{d, g, h\} \times 3,\ \{i\} \times 6)$ and $k = 8$.[10] Hence, in this instance we have $n = 16$ and $n/k = 2$. Consider the committee $W = \{a, b, c, e, f, g, h, i\}$. This committee violates JUQ, and (c, d) is a JUQ swap. However, this change increases overrepresentation with respect to the natural measures discussed above. The number of candidates whose supporters are overrepresented increases: from $\{a, b, c\}$ to $\{e, f, g, h\}$. Similarly, the number of unique overrepresented voters grows from 4 to 6. The sum of the magnitudes of the violations also increases (from a total of 4 points over upper quota to 6).

Aside from the fact that this shows that another approach is required to show that the swaps eventually terminate, this might seem like a problem with JUQ: indeed, an axiom that aims to prevent overrepresentation should not require swaps that increase overrepresentation. However, we would argue that this is not as problematic as it might first seem. Indeed, from the example above, W is the unique committee that minimizes the three measures we mentioned. On the other hand, this committee violates a rather weak notion of efficiency: it includes e but not d, even though $N(e) \subset N(d)$. In other words, including d instead of e would make everyone equally satisfied or better off. So it appears that minimizing overrepresentation (at least in the sense above) is not a reasonable objective, as it clashes with an extremely weak and desirable notion of efficiency.

Still, this shows that we need a different potential function. The proof of Theorem 5 already gives us a possible option: JUQ swaps always increase the Adams-AV-score of a committee (i.e., they increase the lexicographic objective function $(\text{score}_{w_1}(\boldsymbol{A}, W), \text{score}_{w_2}(\boldsymbol{A}, W))$). Since there are only finitely many committees, no infinite sequences can exist.

Proposition 7. *There are no infinite sequences of JUQ swaps.*

As a consequence, every sequence of JUQ swaps has length at most $\binom{m}{k}$, since visiting the same committee twice would lead to the possibility of an infinite sequence. The fact that JUQ swap eventually terminate, however, is still not enough to show polynomial-time computability of JUQ-compliant exhaustive committees as, in principle, one could need super-polynomially many swaps to reach a (locally) optimal committee where no JUQ swaps are possible.[11] We leave a precise

[10]The notation $\{a, b, c\} \times 4$ means that there are four copies of the ballot $\{a, b, c\}$.

[11]For example, for PAV, one might need super-polynomially many swaps to converge [Kraiczy and Elkind, 2024].

characterization of such dynamics for future work. In the next section, we tackle the complexity of constructing JUQ committees from another angle.

3.4 The Upper Quota Elimination Rule

In addition to Thiele rules such as PAV, several sequential rules that aim to prevent underrepresentation have been introduced in the multi-winner voting literature. In particular, we recall the definition of the Greedy Justified Candidate Rule (GJCR) [Brill and Peters, 2023], which satisfies EJR+ (but fails JUQ, see the full paper [Baychkov et al., 2026]).

Definition 9. GJCR begins with the empty committee $W = \emptyset$ and iteratively adds candidates witnessing an EJR+ violation to it. Among all EJR+ violations witnessed by a candidate d and a non-empty set of voters $N' \subseteq N(d)$, GJCR picks the one maximizing $|N'|$ and adds d to W. The rule terminates once no EJR+ violations remain, returning a committee W with $|W| \leq k$. △

We take a similar sequential approach to devise the *Upper Quota Elimination Rule* (UQER), showing that exhaustive committees satisfying JUQ can be found in polynomial time in the process. We take the “dual approach” to GJCR: instead of adding EJR+ witnesses, our rule starts off with the entire set of candidates C and then iteratively deletes candidates witnessing an upper quota violation. This is repeated until either no candidate is overrepresented (and hence the remaining set of candidates satisfies UQ) or until the rule reaches a committee of size k. We will show that this committee and any set considered in between indeed satisfies JUQ.

Formally, our rule proceeds as follows. We start with $W_0 = C$, the set of all candidates. In each step $j = 1, 2, \dots$ we choose a candidate to remove. To do so, we find all candidates $c \in W_{j-1}$ that cause a UQ violation for W_{j-1}, that is $|A_i \cap W_{j-1}| > \lceil q_c \rceil$ for all $i \in N(c)$. Let c_j be the candidate among these with the fewest supporters $|N(c)|$, with ties broken arbitrarily. Set $W_j = W_{j-1} \setminus \{c_j\}$ and repeat. We continue until the first step j^* in which either $|W_{j^*}| = k$ or W_{j^*} satisfies UQ. In the first case, our rule outputs $W = W_{j^*}$, and in the second case it outputs an arbitrary subset $W \subseteq W_{j^*}$ with $|W| = k$. We could, for instance, remove the $|W_{j^*}| - k$ candidates with fewest approvals.[12]

Theorem 8. *UQER returns an exhaustive committee satisfying JUQ in polynomial time.*

To prove this claim we show that JUQ is always maintained during the execution of UQER.

Lemma 9. *Every intermediate set of candidates in $\{W_0, W_1, \dots, W_{j^*}\}$ produced during the execution of UQER satisfies JUQ.*

Proof. It is easy to see that the initial set $W_0 = C$ satisfies JUQ, as there are no candidates outside the set that can be part of a JUQ violation. We can also observe that the set of UQ-violating candidates never increases during the execution of UQER.

Suppose at some point we encounter a JUQ violation in the set W_j, witnessed by some candidate pair (c, d) and voter set $N' \subseteq N(d)$.[13] This means that $|A_i \cap (W_j \setminus \{c\} \cup \{d\})| \leq \lceil q(N') \rceil \leq \lceil q_d \rceil$ for all $i \in N'$, and thus $|A_i \cap (W_j \setminus \{c\})| < \lceil q_d \rceil$.

As $d \notin W_j$ it must have been removed in some earlier step x, i.e. $d = c_x$ for some $x < j$. As d was removed in this step, we know that $|A_i \cap W_{x-1}| > \lceil q_d \rceil$ for all $i \in N(d)$ as d was a UQ-violating candidate for W_{x-1}. Thus, after d was removed, $|A_i \cap W_x| \geq \lceil q_d \rceil$ for all $i \in N(d)$. Further, by our selection of d we know that d was the overrepresented candidate with the least approvals. Hence, $q_d \leq q_{d'}$ for any other candidate d' whose supporters are overrepresented at

[12]We note that UQER is similar to the dual-quota apportionment method of Mayberry [1978]. However, even for apportionment, UQER differs as we explicitly use the committee size k, and restrict our deletions to candidates who are strictly (not weakly) above the upper quota. Further, Mayberry [1978] uses a different method to select which candidates to delete.

[13]Note that c need not be the same as c_{j+1}, i.e., the candidate that will be removed from W_j to obtain W_{j+1}.

step x. Now let $i \in N' \setminus N(c)$ be any voter from the underrepresented group not approving c. Note that by the same argument as in Theorem 5 such a voter must exist. As $|A_i \cap W_x| \geq \lceil q_d \rceil$ and $|A_i \cap W_j| = |A_i \cap W_j \setminus \{c\}| < \lceil q_d \rceil$, there must exist a candidate $c' \in A_i \cap (W_x \setminus W_j)$. Let c' be the last such removed candidate and assume it was removed at a step y between step x and step j. As it was the last, we know that $|A_i \cap (W_{y-1} \setminus \{c'\})| < \lceil q_d \rceil$. By our assumption that d was a candidate with the least approvals when removed at step x we know that $q_{c'} \geq q_d$. In particular, for this step it must hold that $|A_i \cap W_{y-1}| > \lceil q_{c'} \rceil \geq \lceil q_d \rceil$, since the set of overrepresented candidates never increases. This, however is a contradiction to $|A_i \cap (W_{y-1} \setminus \{c'\})| < \lceil q_d \rceil$ and thus (c, d) could not have been a JUQ violation. Thus every candidate set from $\{W_0, W_1, \ldots, W_{j^*}\}$ satisfies JUQ. □

With this, we can easily prove our theorem.

Proof of Theorem 8. Using Lemma 9 we know that W_{j^*} satisfies JUQ. If $|W_{j^*}| = k$ then we are done. Meanwhile, if $|W_{j^*}| > k$ we know that W_{j^*} additionally satisfies UQ. Thus, any subset $W \subseteq W_{j^*}$ will satisfy UQ and thus also the weaker JUQ. Finally, UQER runs in polynomial time, as we only need to check which candidates are overrepresented for at most $m - k$ iterations. □

Similarly to Lemma 9, we can show UQER maintains EJR+ during its execution, until the final step. We will use this result later, in Section 5. Note that the result does not necessarily hold for the arbitrary subset $W \subseteq W_{j^*}$ that is returned if $|W_{j^*}| > k$.

Lemma 10. *Every intermediate set of candidates $\{W_0, W_1, \ldots, W_{j^*}\}$ produced during the execution of UQER satisfies EJR+.*

4 Balancing Under- and Overrepresentation

We now turn to our second view on overrepresentation: near quota. Intuitively, there should not exist a party P_i which is overrepresented and a party P_j which is underrepresented such that by moving a seat from party P_i to party P_j we can bring both parties closer to their respective quotas. Our goal is to generalize this to the multi-winner voting setting, by again interpreting single candidates as parties that represent their voters. In this spirit, there should not exist a candidate inside the committee whose approvers are overrepresented and a candidate outside the committee for whom a subset of the approvers are underrepresented such that we could swap these two candidates and make both "better represented". To better interpret this, we again return to the apportionment setting. We first give the following equivalent formulation of apportionment near quota: two parties P_i and P_j witnessing a near quota violation is equivalent to the first party being more than 0.5 above its quota, and the second party being more than 0.5 below its quota.

Observation 11. *An outcome W in an apportionment instance satisfies apportionment near quota if and only if there are no two parties P_i and P_j such that $a_i(W) - q(P_i) > \frac{1}{2}$ and $q(P_j) - a_j(W) > \frac{1}{2}$.*

To generalize this to the approval setting, we have to account for potentially overlapping approval sets. Assume that $c \in W$ is the overrepresented candidate and $d \in C \setminus W$ is the underrepresented one and let $N' \subseteq N(d)$ be the set of underrepresented voters approving d. Then we say that c and d witness a *justified near quota* violation if swapping c with d moves every voter in $N(c)$ closer to their quota q_c *whenever their satisfaction changes* (i.e., when they do not also approve of d) and every voter in N' moves closer to their quota $q(N')$ whenever their satisfaction changes. For voters approving of both of c and d the satisfaction does not change. For these voters we merely require that they are above the quota q_c and below the quota $q(N')$ to ensure these voters are not on the "wrong side" of their respective quotas.

In the following, $\mathbf{1}[\cdot]$ denotes the indicator function, i.e., $\mathbf{1}[p] = 1$ if p is true and 0 otherwise.

Definition 10. A set W satisfies *justified near quota* (JNQ) if there does not exist a pair of candidates $c \in W$ and $d \in C \setminus W$, and non-empty set of voters $N' \subseteq N(d)$ such that

$$|A_i \cap W| - q_c > \frac{1}{2} \cdot \mathbf{1}[d \notin A_i] \qquad \text{for all } i \in N(c), \text{ and}$$
$$q(N') - |A_i \cap W| > \frac{1}{2} \cdot \mathbf{1}[c \notin A_i] \qquad \text{for all } i \in N'. \qquad \triangle$$

The following example shows that, as one would expect, JNQ and JUQ are two distinct notions.

Example 6. Consider the profile $\boldsymbol{A} = (\{a\}, \{a\}, \{a\}, \{a, c\}, \{b\})$ and let $k = 2$, so the quota of a single voter is $2/5$. The committee $\{a, c\}$ satisfies JNQ. Indeed, replacing either a or c with b would change the satisfaction of the voter submitting $\{b\}$ from 0 to 1, increasing the distance from their quota of $2/5$. On the other hand, this committee fails JUQ as the voter approving $\{a, c\}$ is overrepresented and we could swap c with b. Conversely, the committee $\{b, c\}$ satisfies JUQ, but fails JNQ. Indeed, here we should swap b with a, and the set $N' \subseteq N(a)$ can consist for example of voters 1 and 2. Finally, the committee $\{a, b\}$ satisfies both axioms. $\triangle$

Indeed, following Observation 11 we can see that JNQ generalizes apportionment near quota.

Proposition 12. *For apportionment instances, JNQ is equivalent to apportionment near quota.*

Next, similarly to Adams-AV and JUQ, we get that every refinement of SLAV satisfies JNQ. The proof is analogous to the Adams-AV proof. That is, we show that if a committee W does not satisfy JNQ as witnessed by two candidates c and d then swapping c with d increases the SLAV score.

Theorem 13. *Any refinement of SLAV satisfies justified near quota.*

Using this, we again obtain a characterization, this time of refinements of SLAV.

Theorem 14. *A composite Thiele rule satisfies apportionment near quota if and only if it is a refinement of SLAV.*

Going beyond Thiele rules, in the full paper [Baychkov et al., 2026] we detail counterexamples that show that essentially all familiar voting rules fail JNQ (e.g., the Method of Equal Shares, Phragmén's rule, and all the rules in the `abcvoting` [Lackner et al., 2023] Python package at the time of writing). In particular, the variance-Phragmén voting rule (which extends Sainte-Laguë's method) also fails JNQ. As an immediate consequence of Theorem 13, analogously to JUQ swaps, we get that there are no infinite sequences of JNQ swaps, as any swap increases the SLAV-score.

Corollary 15. *There are no infinite sequences of JNQ swaps.*

Finally, similarly to JUQ and EJR+, we show that it can be verified in polynomial-time whether a given committee satisfies JNQ.

Proposition 16. *Whether a committee satisfies JNQ is decidable in polynomial time.*

It is open whether exhaustive committees satisfying JNQ can be computed in polynomial time.

5 Combining Lower, Near, and Upper Quota Axioms

In this section, we investigate whether our upper, near, and lower quota axioms can be simultaneously satisfied in the multi-winner voting setting.

First, we observe that, as a refinement of CC, Adams-AV satisfies the rather weak underrepresentation axiom of *justified representation* [Aziz et al., 2017], in addition to satisfying JUQ and being exhaustive. Can we do better? In particular, is it always possible to find an (exhaustive) rule that satisfies EJR+ and JUQ, or maybe even EJR+, JUQ and JNQ?

Unfortunately, in the class of Thiele rules, this is not the case. We have seen a "tripartition" with regard to representation axioms. PAV, and its refinements, are the only composite Thiele rules satisfying EJR+, SLAV refinements are the only ones satisfying JNQ, and Adams-AV refinements are the only ones satisfying JUQ, with these (im)possibilities mirroring the apportionment setting.

For the apportionment setting, however, Balinski and Young [2001, Proposition 6.5] prove a curious fact about divisor methods: any committee selected by a divisor method either satisfies lower quota or upper quota. Interestingly enough, we can generalize this result to the approval-based multi-winner voting setting and show that for a large class of Thiele rules (corresponding to the divisor methods as defined by Balinski and Young) including PAV and SLAV, every selected committee satisfies either UQ or EJR+.

Theorem 17. *Let $w = (w(1), w(2), \dots)$ be a weight function with strictly positive weights for which there exists an $\alpha \in \mathbb{R}_{>0}$ satisfying $(i-1) < \frac{\alpha}{w(i)} \le i$ for all $i \in \mathbb{N}_{>0}$ or $(i-1) \le \frac{\alpha}{w(i)} < i$ for all $i \in \mathbb{N}_{>0}$. Then any committee selected by the simple w-Thiele rule satisfies either UQ or EJR+.*

Proof. Assume that W is selected by w-Thiele and that it fails both UQ and EJR+. Let c and d be the respective witnesses of this. Thus, it holds that $|A_i \cap W| > \lceil q_c \rceil$ for every $i \in N(c)$ as well as $|A_i \cap W| < \lfloor q(N') \rfloor$ for some non-empty $N' \subseteq N(d)$ and every $i \in N'$.

Now consider the score change by exchanging c with d. The score of every voter $i \in N' \setminus N(c)$ increases by $w(|A_i \cap W| + 1)$, while the score of every $i \in N(c) \setminus N'$ decreases by $w(|A_i \cap W|)$. With this we obtain

$$\begin{aligned}\sum_{i \in N' \setminus N(c)} w(|A_i \cap W| + 1) - \sum_{i \in N(c) \setminus N'} w(|A_i \cap W|) &\ge \sum_{i \in N'} w(|A_i \cap W| + 1) - \sum_{i \in N(c)} w(|A_i \cap W|) \\ &\ge \sum_{i \in N'} w(\lfloor q(N') \rfloor) - \sum_{i \in N(c)} w(\lceil q_c \rceil + 1) > |N'| \frac{\alpha}{q(N')} - n_c \frac{\alpha}{q_c} = \alpha(\frac{n}{k} - \frac{n}{k}) = 0.\end{aligned}$$

Hence, by swapping c with d the score would increase and thus W was not an optimal committee according to w-Thiele. □

As a corollary, we immediately obtain that this is true for PAV, SLAV, but also Adams-AV.

Corollary 18. *Any committee selected by PAV, SLAV, or Adams-AV satisfies either UQ or EJR+.*

Proof. For PAV we observe that it satisfies the definition with $\alpha = 1$ and SLAV with $\alpha = \frac{1}{2}$. For Adams-AV we observe that, for a fixed instance with n voters and target committee size k, Adams-AV is equivalent to the simple Thiele rule with vector $(n \cdot k, 1, \frac{1}{2}, \dots)$ (we prove this fact in the full paper [Baychkov et al., 2026]), which satisfies the condition with $\alpha = 1$. □

As another consequence, these rules satisfy EJR+ on instances in which no UQ committee exists. The aforementioned classical result of Balinski and Young [2001] was partially generalized for a restricted class of divisor methods by Cembrano et al. [2025], who showed that for any fixed apportionment instance there exists a divisor method satisfying both upper and lower quota in that instance. An interesting open question is whether this also generalizes to multi-winner voting: for any instance does there exist a Thiele method that satisfies EJR+ and JUQ in that instance?

If we consider arbitrary multi-winner voting rules, we can show that UQ (and thus JUQ), JNQ and EJR+ are jointly satisfiable, as long as we do not require exhaustiveness. To this end, we define a voting rule, the *Proportional Elimination Rule*, in a similar way to UQER, but with a different initial committee and a modified method of selecting candidates for removal.

We start from any committee W_0 output by GJCR (Definition 9). This committee has size $|W_0| \le k$. In each step $j \in \{1, 2, \dots\}$, we find all candidates $c \in W_{j-1}$ for which all supporters $i \in N(c)$ are strictly above their lower quota, i.e. $|A_i \cap W_{j-1}| > \lfloor q_c \rfloor$.[14] If there are no such candidates then we are done and our voting rule outputs $W^* = W_{j-1}$. Otherwise, let c_j be the candidate from among these with the least approvals (with ties broken arbitrarily), set $W_j = W_{j-1} \setminus \{c_j\}$, and repeat.

Proposition 19. *The (possibly not exhaustive) committee W produced by the Proportional Elimination Rule satisfies UQ, JNQ and EJR+.*

With exhaustiveness, the picture gets more complicated. Indeed, it remains an open problem whether any of the three pairs (JUQ, JNQ), (JUQ, EJR+) and (JNQ, EJR+) are jointly satisfiable for exhaustive committees. In the following, we explore the interplay of JUQ, EJR+ and exhaustiveness in more detail. For other notions of proportionality, however, we can give a clear answer. We show in the full version of the paper [Baychkov et al., 2026] that JUQ is incompatible with several other popular axioms for multi-winner voting, namely priceability, fully justified representation, and perfect representation. The latter two impossibilities hold even without requiring exhaustiveness.

Our first result shows that JUQ, EJR+ and exhaustiveness are "almost" compatible, i.e., we can either find an exhaustive committee satisfying EJR+ and JUQ or we can always find two sets of candidates satisfying both EJR+ and UQ, one with $\leq k$ candidates and one with $\geq k$ many.

Corollary 20. *There always exists a committee $W_{\leq}$ satisfying EJR+ and UQ with $|W_{\leq}| \leq k$. Additionally, one of the following is always true:*

- *There exists a committee $W_{=}$ satisfying EJR+ and JUQ with size exactly $|W_{=}| = k$.*
- *There exists a set of candidates $W_{\geq}$ satisfying EJR+ and UQ with size $|W_{\geq}| \geq k$.*

Proof. $W_{\leq}$ is output by the Proportional Elimination Rule. To find either $W_{=}$ or $W_{\geq}$, recall from Theorem 8 and Lemma 10 that during the execution of UQER we find a candidate set W_{j^*} that satisfies EJR+ and JUQ and, additionally, either (1) has $|W_{j^*}| = k$, giving us $W_{=} = W_{j^*}$, or (2) satisfies UQ and has $|W_{j^*}| > k$, giving us $W_{\geq} = W_{j^*}$. □

A voting rule satisfying JUQ, EJR+ and exhaustiveness is unlikely to be as straightforward as some of the voting rules we proposed earlier. One might be tempted to modify the outcome of a rule that satisfies two of EJR+, JUQ, and exhaustiveness to also satisfy the third. Below we provide negative results for three of the most natural modifications. First, we show that the Proportional Elimination Rule cannot be made exhaustive (while satisfying JUQ) by adding additional candidates.

Proposition 21. *There exists an instance where every exhaustive superset of the outcome of the Proportional Elimination Rule fails JUQ.*

Proof. Consider the instance with $n = 63$ and $k = 7$ depicted in Table 2. Here, GJCR selects only the non-exhaustive committee $W = \{c_1, c_2, c_3, c_4, c_5\}$. Notice that, for every $c \in W$ and $i \in N(c)$, we have that $|A_i \cap W| \leq \lfloor q_c \rfloor$. Thus the Proportional Elimination Rule also selects W.

Now consider any possible completion of this committee to 7 candidates. Without loss of generality, let us look at $W' = \{c_1, \ldots, c_7\}$. Then c_5 and c_8 witness a JUQ violation with $N' = N(c_8)$. Indeed, $q_{c_5} = 1$ and every voter $i \in N(c_5)$ has $|A_i \cap W'| = 2$. Moreover, $\lceil q_{c_8} \rceil = 3$ and every voter $i \in N(c_8)$ has $|A_i \cap (W' \setminus \{c_5\} \cup \{c_8\})| \leq 3$. □

In other words, we cannot always start with $W_{\leq}$ from Corollary 20 and add $k - |W_{\leq}|$ candidates to this committee while continuing to satisfy EJR+. Analogously, we cannot always remove $|W_{\geq}| - k$ candidates from $W_{\geq}$ from Corollary 20 while continuing to satisfy JUQ.

Proposition 22. *The partial outcome W_{j^*} of UQER (Section 3.4) need not contain a size k subset that satisfies EJR+.*

[14] Note that this is different to UQER, which removed candidates violating UQ, i.e. those with all supporters above their upper quota.

Finally, we consider whether we can start from an exhaustive committee satisfying EJR+, and modify it by *cleverly* choosing JUQ swaps, such that EJR+ is maintained. Unfortunately, this is not always possible. Consider a multi-winner election with $k = 5$ and $\boldsymbol{A} = (\{a, b\} \times 3,\ \{a, b, c\},\ \{c, d, e\},\ \{d, e\} \times 3,\ \{f\},\ \{g\})$. Committee $W = \{a, c, d, f, g\}$ satisfies EJR+. However c witnesses a JUQ violation with either b and voter set $N(b)$ or e and voter set $N(e)$. These are the only JUQ violations, and satisfying either would lead to a committee that fails EJR+. This claim holds even if we select a particularly well chosen EJR+ committee, such as the output of the Method of Equal Shares. In the interest of space, we defer the definition of this rule to the full version of the paper [Baychkov et al., 2026].

Candidate	3	3	3	14	7	7	14	12
c_1, c_2				✓	✓			✓
c_3, c_4						✓	✓	
c_5	✓	✓	✓					
c_6	✓	✓		✓				
c_7	✓		✓		✓	✓		
c_8		✓	✓				✓	

Table 2. Example instance for Proposition 21 with $k = 7$. The column headers refer to the number of voters submitting that ballot (e.g., there are 3 voters approving $\{c_5, c_6, c_7\}$).

Proposition 23. *A sequence of JUQ swaps that starts from a committee satisfying EJR+ might terminate with a committee that violates EJR+. This holds even if the initial committee was selected by the Method of Equal Shares (with any completion method), and even if we use the profile of a candidate's supporter satisfactions*[15] *to select which JUQ swap to perform next.*

6 Conclusion and Research Directions

In this paper, we studied overrepresentation in approval-based multi-winner voting. We defined an axiom aimed at avoiding overrepresentation whenever possible, called justified upper quota (JUQ). We further introduced the class of composite Thiele rules and showed that, among this class, Adams-AV and its refinements are the only rules satisfying JUQ. Moreover, we showed that JUQ is satisfiable in polynomial time via the Upper Quota Elimination Rule. Additionally, we introduced an axiom that aims at balancing under- and overrepresentation, justified near quota (JNQ), and showed that, among the class of composite Thiele rules, only SLAV and its refinements satisfy it. Finally, we discussed the relationship between our notions and established lower quota notions, such as EJR+. We note that some of our results, such as the (partial) characterization of Adams-AV and SLAV through JUQ and JNQ (respectively), mirror and extend those obtained in the classical apportionment literature [Balinski and Young, 2001].

We identify several interesting open questions for future work. A first task would be to characterize the relationship between JUQ and JNQ for exhaustive committees and, similarly, the relationship between our axioms and EJR+. In other words, does there exist an exhaustive rule that avoids under- and overrepresentation at the same time? We know from our results that this is not possible within the composite Thiele class and that such a rule must fail the commonly studied axiom of priceability. Next, we note that all other established voting rules we have examined fail both our axioms. This suggests that our axioms embody a rather different approach to multi-winner voting rules compared to the existing literature. Therefore, we believe that finding natural and normatively-appealing rules that satisfy our axioms could lead to fundamentally new insights in multi-winner voting. Finally, we point out that approval voting is only one possible ballot type in multi-winner voting. It would be interesting to explore overrepresentation for ordinal or cardinal preferences or even for subset selection problems beyond multi-winner voting, such as recently done for clustering by Jerrett and Anshelevich [2025].

[15] For some candidate $c \in W$ this profile would consist of the multiset $\{|A_i \cap W|\}_{i \in N(c)}$.

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